

Review of Cross-Product (Sec. 12.4) (3-1)

① Definition

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note: ~~matrix~~
cross-product is
a vector!!!

MUST know Eq. 6
on p. 856 (9th Ed.)

② Main properties of cross-product

1) $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

2) Corollary of 1):

$\vec{u} \times \vec{u} = \vec{0}$. More generally:

$\vec{u} \times (k\vec{u}) = k \cdot (\vec{u} \times \vec{u}) = \vec{0}$.
any scalar

But $k\vec{u} \parallel \vec{u}$ (Sec. 12.2). Thus:

$$(\vec{u} \times \vec{v} = \vec{0}) \Leftrightarrow (\vec{u} \parallel \vec{v})$$

- This formula can be used to check if $\vec{u} \parallel \vec{v}$ when we don't "see" components of $\vec{u}$ & $\vec{v}$ as numbers.
- (when we see them as numbers, we just do an inspection if $\vec{v} = k \cdot \vec{u}$ for some k).

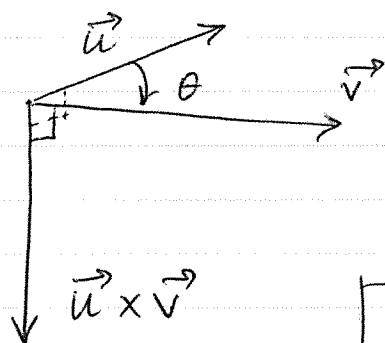
think of this as $k\vec{u}$.

→ Other properties: **must** see p. 859 (e-book).
(9th Ed.)

③ Geometric meaning & purpose of cross-product

3-2

direction:
"right-hand rule"

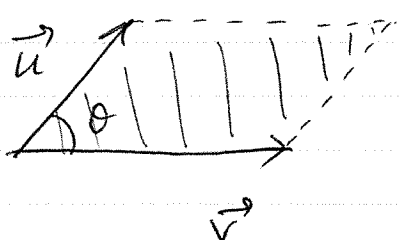


$$\begin{aligned} (\vec{u} \times \vec{v}) &\perp \vec{u} \\ (\vec{u} \times \vec{v}) &\perp \vec{v} \end{aligned}$$

Thus,

$$(\vec{u} \times \vec{v}) \perp \left(\text{plane made by } \vec{u} \text{ \& } \vec{v} \right)$$

The main use of cross-product:
it is a vector that is $\perp$ to two given vectors.



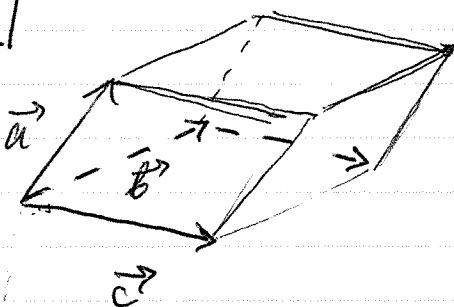
$$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin \theta$$

(magnitude of $\vec{u} \times \vec{v}$)

(area of parallelogram made by $\vec{u}$ & $\vec{v}$.)

④ Triple product

[a]



$$V = \left| \vec{a} \cdot (\vec{b} \times \vec{c}) \right|$$

↑
volume of
slanted box
made by
 $\vec{a}, \vec{b}, \vec{c}$

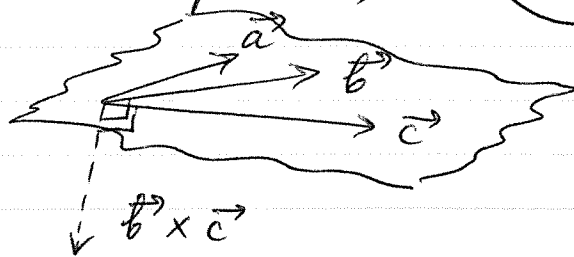
in any
permutation

Formula: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

[b] If $\vec{a}, \vec{b}, \vec{c}$ lie in same plane,
then $V = 0$;

(3-3)

(We can also see
that in this case
 $\vec{a} \perp (\vec{b} \times \vec{c})$,



$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c})$ must $= 0 \leftarrow$ see Sec. 12.3.)

Thus:

$$\boxed{(\vec{a}, \vec{b}, \vec{c} \text{ lie in same plane}) \Leftrightarrow (\vec{a} \cdot (\vec{b} \times \vec{c}) = 0)}$$

[See Ex. 5 in book for numbers.]

[c] In Lecture 1, topic (2c) we showed:

$$\left(\begin{array}{l} \vec{a}, \vec{b}, \vec{c} \text{ lie in same plane} \\ \text{(and } \vec{a} \nparallel \vec{b}) \end{array} \right) \Rightarrow \left(\vec{c} = s\vec{a} + t\vec{b} \right) \text{ for some scalars } s, t$$

Now we can show $\Leftarrow$.

Proof: Let $\vec{c} = s\vec{a} + t\vec{b}$.

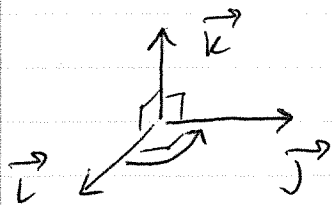
To test if $\vec{c}$ is in same plane
with $\vec{a}$ & $\vec{b}$, compute triple prod:

$$\begin{aligned} \vec{c} \cdot (\vec{a} \times \vec{b}) &= (s\vec{a} + t\vec{b}) \cdot (\vec{a} \times \vec{b}) \\ &= s \cdot \underbrace{\vec{a} \cdot (\vec{a} \times \vec{b})}_{\perp \vec{a}} + t \cdot \underbrace{\vec{b} \cdot (\vec{a} \times \vec{b})}_{\perp \vec{b}} \\ &= s \cdot 0 + t \cdot 0 = 0. \quad \checkmark \end{aligned}$$

So, $\vec{c}$ is in same plane as $\vec{a}, \vec{b}$. q.e.d.

⑤ Cross-product of $\vec{i}$, $\vec{j}$, $\vec{k}$.

3-4



$$|\vec{i} \times \vec{j}| = |\vec{i}| \cdot |\vec{j}| \cdot \sin \theta = 1.$$

$\swarrow \quad \swarrow \quad \swarrow$
 $1 \quad 1 \quad \sin 90^\circ = 1.$

Direction: $(\vec{i} \times \vec{j}) \perp xy\text{-plane}$,
points up.

Thus,

$$\begin{aligned}\vec{i} \times \vec{j} &= \vec{k} \\ \vec{j} \times \vec{k} &= \vec{i} \\ \vec{k} \times \vec{i} &= \vec{j}\end{aligned}$$

 (so $\vec{i} \times \vec{k} = -\vec{j}$)

Also, $\vec{i} \times \vec{i} = \vec{0} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k}$.
(see topic ② above).