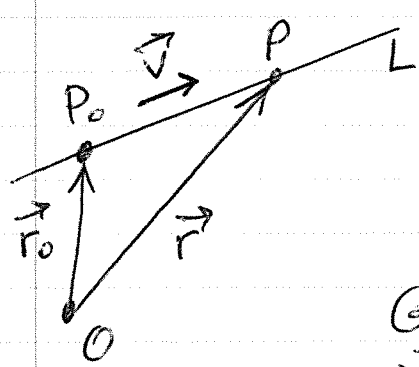
Review of Straight lines in 3D. (Sec. 12.5)

① Vector & parametric eqs of line



$\vec{r} = \langle x, y, z \rangle$

notation that will be extensively used in this course.

Given:

- point P_0 on L
- vect. $\vec{v} \parallel L$

Vector eq. of L:

$$\boxed{\vec{r}(t) = \vec{r}_0 + \vec{v}\,t}$$

By components:

$$\left.\begin{aligned} x &= x_0 + a t \\ y &= y_0 + b t \\ z &= z_0 + c t \end{aligned}\right\} \text{parametric form}$$

Analogy:
Uniform motion of a particle on a line.
$\boxed{a = \text{speed}, \quad t = \text{time}.}$

② "Ingredients" of a line

("given a line") $\overset{}{\Longleftrightarrow}$ means (we are given its $\boxed{\text{two ingredients}}$:
- point P_0 on L;
- vector $\vec{v}$ along L.)

In detail:

"$\Longrightarrow$" If we "have a line": then we automatically know point $(x_0, y_0, z_0) = P_0$; and vector $\vec{v} = \langle a, b, c \rangle$.

$$\begin{aligned} x &= \overbrace{\boxed{x_0}}^{P_0} + \overbrace{\boxed{a}}^{\vec{v}}\,t \\ y &= \boxed{y_0} + \boxed{b}\,t \\ z &= \boxed{z_0} + \boxed{c}\,t \end{aligned}$$

"$\Leftarrow$" Conversely, if we know P_0 & $\vec{v}$, we know the eqs. of the line (topic ①). (4-2)

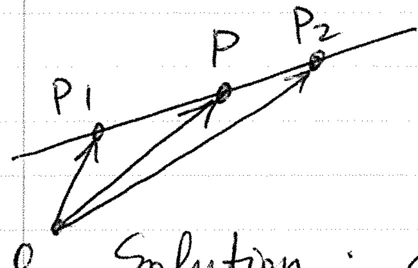
③ Symmetric eqs. of line

$$\left.\begin{aligned} x - x_0 &= at \\ y - y_0 &= bt \\ z - z_0 &= ct \end{aligned}\right\} \Rightarrow t = \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Note: Again, given symmetric eqs. of a line, we automatically are given:

- point $P_0 = (x_0, y_0, z_0)$
- vector $\vec{v} = \langle a, b, c \rangle$ along L.

④ Line through two given points.



Given: $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$

Find: line P_1P_2.

Solution: The main idea is to reduce any new problem about a line to its form in topic ①.

That is, we need to find two ingredients of line P_1P_2:

- point on it; $\longrightarrow P_1$
- vector along it: $\longrightarrow \overrightarrow{P_1P_2}$.

Thus, answer: $\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P_2} \cdot t$.

See Ex. 2(a) in book for numbers.

⑤ Segment connecting two points

Q: What is the meaning of t in the previous equation?

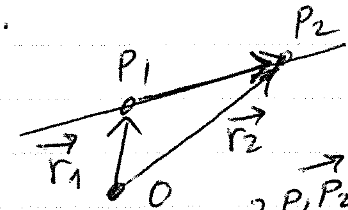
$t=0$: $\vec{OP}(t=0) = \vec{OP_1} + \cancel{\vec{P_1P_2}\cdot 0} = \vec{OP_1}$

$\Rightarrow P(t=0) = P_1$

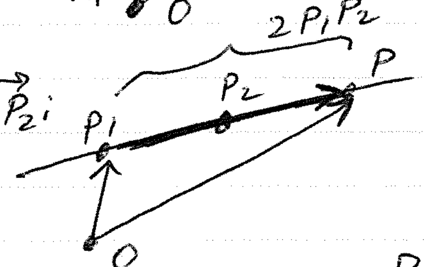
$t=1$: $\vec{OP}(t=1) = \vec{OP_1} + \vec{P_1P_2}\cdot 1 = \vec{OP_1} + \vec{P_1P_2}$
$= \vec{OP_2}$

see the Side Note below

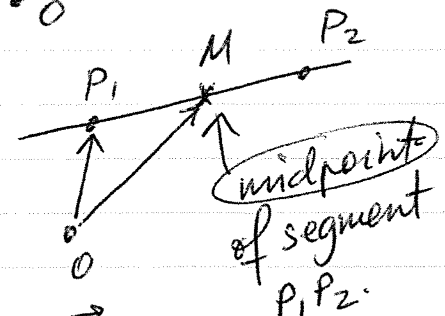
$\Rightarrow P(t=1) = P_2$.



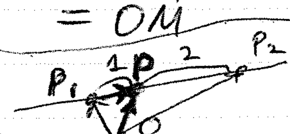
$t=2$: $\vec{OP}(t=2) = \vec{OP_1} + 2\vec{P_1P_2}$:



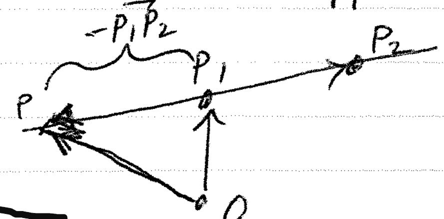
$t=\frac{1}{2}$: $\vec{OP}(t=\frac{1}{2}) = \vec{OP_1} + \frac{1}{2}\vec{P_1P_2}$
$= \vec{OM}$



$t=\frac{1}{3}$: ⟶



$t=-1$: $\vec{OP}(t=-1) = \vec{OP_1} + (-1)\vec{P_1P_2}$:



Side note: $\vec{OP_2} = \vec{OP_1} + \vec{P_1P_2}$

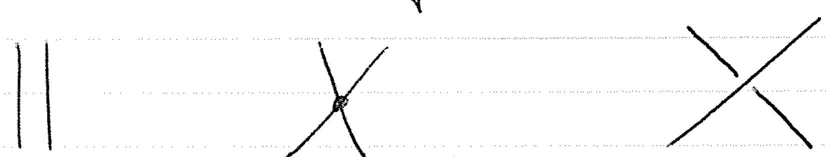
$\Rightarrow \vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = \vec{r_2} - \vec{r_1}$

Conclusion:

- $\boxed{0 \le t \le 1}$, eq. $\boxed{\vec{r}(t) = \vec{r}_1 + \overbrace{(\vec{r}_2 - \vec{r}_1)}^{\text{see sidenote}} \cdot t}$
defines segment P_1P_2 (i.e. all pts between P_1 & P_2).

- $t < 0$ or $t > 1$; same eq. defines points on the line P_1P_2 but outside the segment.

⑥ Relative location of two lines in $\mathbb{R}^3$

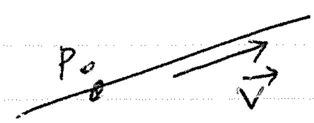


parallel   intersecting   skew

See Ex. 3 in book.

⑦ General comments

[a]



The same line can have many different (but equivalent!) parametric eqs, because: P_0 is not unique ;
$\vec{v}$ is not unique (up to a scalar multiple: $2\vec{v}$, $3\vec{v}$, etc.)

[b] How many eqs. do we need to define a line?

In 2D need 1 eq: $y = kx + m$
(or $ax + by + c = 0$).

In 3D, $\boxed{\text{need 2 eqs:}}$!!

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Why?

0 eqs. $\rightarrow$ R^3 (no constraints on x, y, z).

1 eq. $\rightarrow$ restrict one degree of freedom (say, $z=0$) $\Rightarrow$ get a plane, $\mathbb{R}^2$.
$(3-1=2)$

2 eqs. $\rightarrow$ restrict one more degree of freedom in that plane, $\Rightarrow$ get a line: $\mathbb{R}^1$.

$$\boxed{\underbrace{3}_{\text{degrees of freedom in } \mathbb{R}^3} - \underbrace{2}_{\text{eqs.} = \text{constraints}} = \underbrace{1}_{\text{unrestricted degree of freedom on the line}}}$$