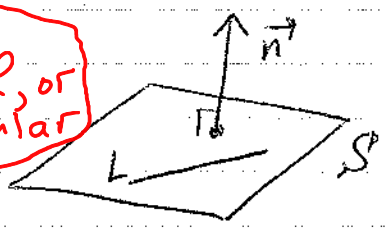
Review of Planes in 3D (sec. 12.5) (5-1)

(1) Equation of a plane. "Ingredients" of a plane

- Recall an intuitive fact:

If $\vec{n}$ is the normal vector to plane S, then $\vec{n} \perp$ to any line L in S.

Means: orthogonal, or perpendicular

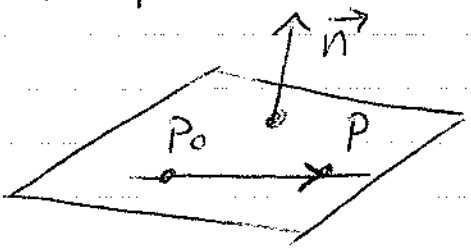


Problem: **Given:**
- point $P_0(x_0, y_0, z_0)$ in S;
- $\vec{n} = \langle a, b, c \rangle$ normal to S.

} "Ingredients" of plane !!!

Find: Eq. of S.

Solution:



Take any pt. P in S. Then line $P_0P \perp \vec{n}$, or $\vec{P_0P} \perp \vec{n}$:

$$\vec{P_0P} \cdot \vec{n} = 0 \Rightarrow \langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$\Rightarrow \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0} \quad (\star)$$

or

$$\boxed{ax + by + cz = d} \quad (\star\star)$$

Thus:

$$\boxed{\left(\text{given the eq. of plane, } (\star) \text{ or } (\star\star)\right) \Leftrightarrow \left(\text{given two "ingredients", } P_0 \text{ \& } \vec{n}\right)}$$

Note: If you are "given" (★), then coord's of one point in that plane are "read off" automatically: (x_0, y_0, z_0).

If you are given (★★), then you can determine a point in S, by, e.g.:

Set $x = 0 = y$; then

$a \cdot 0 + b \cdot 0 + c \cdot z = d \Rightarrow z = d/c$,

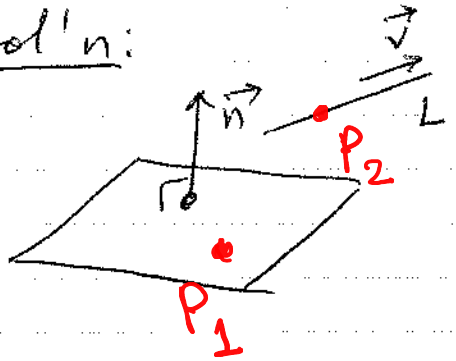
$\Rightarrow P = (0, 0, d/c)$.

Obviously, this is one out of ∞ many points.

Ex. 1 Determine if the given line & plane are parallel:

$S: 2x - y + z = 3$; $L: \begin{aligned} x &= 3t \\ y &= 5t \\ z &= -t + 4 \end{aligned}$

Sol'n:



0) SKETCH, including all the ingredients.

1) $L \parallel S$ iff $\vec{v} \perp \vec{n}$.

Thus, we need to read off $\vec{v}$ and $\vec{n}$ from the given eqs.

2) $\vec{v} = \langle 3, 5, -1 \rangle$; $\vec{n} = \langle 2, -1, 1 \rangle$.

3) It remains to check:

$\vec{v} \circ \vec{n} = \langle 3, 5, -1 \rangle \circ \langle 2, -1, 1 \rangle \overset{\checkmark}{=} 0$

Yes, $L \parallel S$. ∎

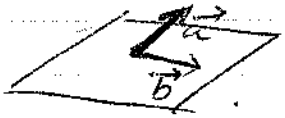
(2) Why do we need 1 eq. for a plane?

0 eqs $\longrightarrow$ $\mathbb{R}^3$ (no constraints on x, y, z).

1 eq. $\longrightarrow$ one variable is constrained, $\Rightarrow$ two variables remain free $\Rightarrow$ $\mathbb{R}^2$.

$$\boxed{3 - 1 = 2}$$

- 3: degrees of freedom in $\mathbb{R}^3$
- 1: constrain 1 variable
- 2: 2 variables define a plane.

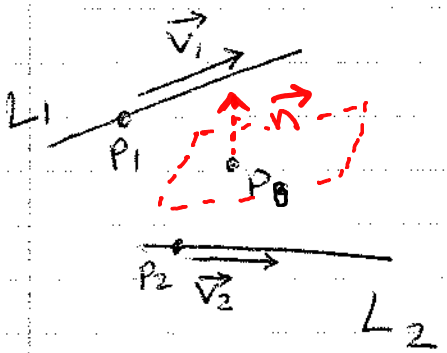


(3) Some other ways to define a plane

Main point: reduce each new problem to the standard form / ingredients considered in topic (1).

(Same idea as in topic (4) of lec. 12.5 - A.)

(a) Plane through a pt. and // to two given lines



Sol'n: 1) List given ingredients in your sketch.

(Dashed lines: things that you need to find.)

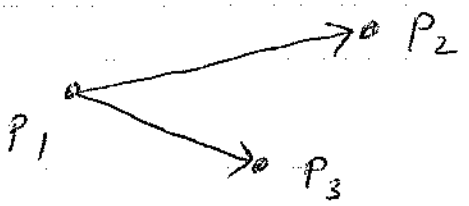
2) For plane S, we have
- ⊕ point P_0. ✓
- ⊖ still need $\vec{n}$.

3) We want $S \parallel L_1 \Rightarrow \vec{n} \perp \vec{v}_1$ (by Ex. 1)
$S \parallel L_2 \Rightarrow \vec{n} \perp \vec{v}_2$.

$\therefore (\vec{n} \perp \vec{v}_1 \text{ & } \vec{v}_2) \Rightarrow (\vec{n} = \vec{v}_1 \times \vec{v}_2)$.

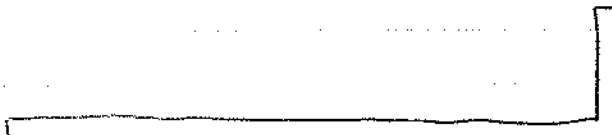
Thus, we have both ingredients for S, $\Rightarrow$ know its equation.

(b) Plane through three points



Given: P_1, P_2, P_3

Find: S, i.e.:
P_1 → a pt. in S;
$\vec{n}$.

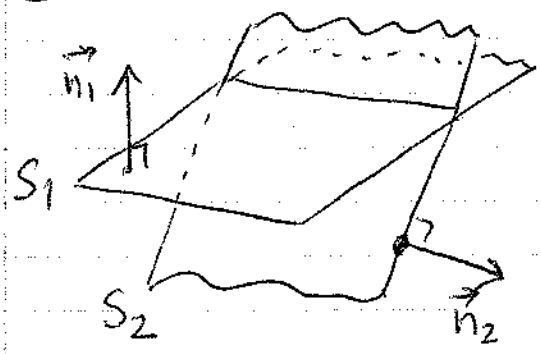


contains $\vec{P_1P_2}, \vec{P_1P_3}$
$\Rightarrow$ is $\perp$ to these vectors.

Thus by (a), $\vec{n} = \vec{P_1P_2} \times \vec{P_1P_3}$.

See Ex. 5 in book for numbers.

(4) Angle between planes $= \angle(\vec{n_1}, \vec{n_2})$.



and by convention it is taken to be $\leq 90°$
(if it is $> 90°$, simply change the direction of $\vec{n_2}$ to opposite).

See topic ① in Lec. 1, 2, 3 and Ex. 7(a) in book.
(dot product) (Sec. 12.5)

(5) Intersection between line & plane

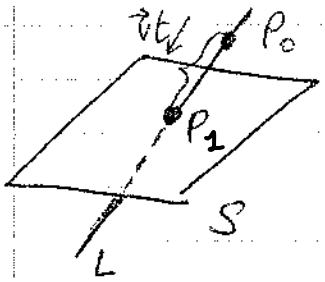
Ex. 2 (extends Ex. 6 in book)

Find the point where line $L: \frac{x-2}{3} = \frac{y}{-4} = \frac{z-5}{1}$ intersects plane $S: 4x+5y-2z=18$.

Sol'n: For this type of problems it is most convenient to use parametric eqs. of L:

$$L: x = 2+3t,\ y = 0-4t,\ z = 5+t. \quad (*)$$

Recall the meaning of parameter t:
- it indicates the location of a point on L (e.g., as at different times we find the point at different locations).



So, we just need to find the "time" t at which the point on the line will cross the plane.

For this, substitute $(*)$ into the eq. of S:

$$4\cdot\underbrace{(2+3t)}_{x} + 5\cdot\underbrace{(0-4t)}_{y} - 2\underbrace{(5+t)}_{z} = 18$$

Solve this eq. for t: $t = -2$.

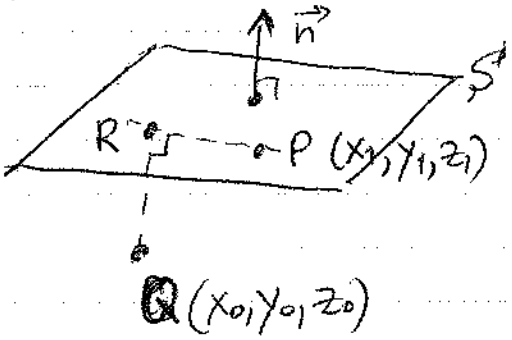
Substitute back into $(*)$:

$$x_1 = 2+3\cdot(-2) = -4$$
$$y_1 = -4\cdot(-2) = 8$$
$$z_1 = 5+(-2) = 3$$

coord's of the intersection point P_1

(6) Distance problems involving planes (5-6)

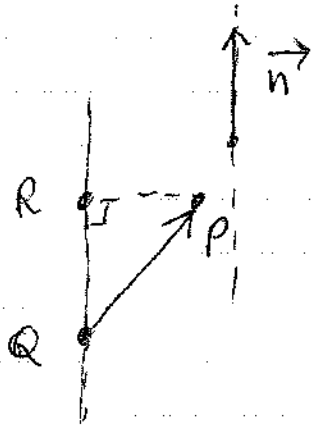
(a) Distance between a pt. & plane



Given: $S \Leftrightarrow$
- P in S $\hookrightarrow (x_1, y_1, z_1)$
- $\vec{n} \perp S$

Find: $|QR|$, where $QR \perp S$

Solution:



$$\vec{QR} = \text{proj}_{\vec{n}} \vec{QP}$$

$$|\vec{QR}| = \text{comp}_{\vec{n}} \vec{QP}$$

$$\vec{QP} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$

Substitute this into formula for comp (topic (4a) in Lec. 2) + use eq. for S.

Details of calculation are in Ex. 8 in book.

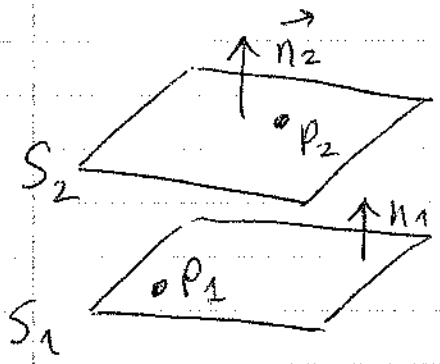
Result:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

distance between plane $ax + by + cz + d = 0$ and pt. (x_0, y_0, z_0).

[6] Distance between two // planes (5-7)

Main point: reduce this new problem to the previous one (just as in topics (3b) & (3a) above).



Given: S_1 & S_2 ⇕ (list ingredients!) $(P_1, \vec{n_1}\,;\ P_2, \vec{n_2})$

Find: $dist(S_1, S_2)$.

Sol'n: this distance is, obviously, that between P_1 & S_2 (or vice versa); so the problem reduces to that in topic (6a). ✓

See Ex. 9 in book.