

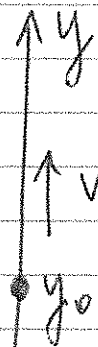
MATH 230

Elementary Differential Equations

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Lecture 1. Introduction① MotivationEx. 1

a) An object starts at y_0 and moves along y -axis with speed v .
Find location $y(t)$ at any t .



Sol'n: $y(t) = y_0 + vt$.

b) Q: What if $v \neq \text{const}$? (1)

We know: $y'(t) = v(t)$ ← integrate

$$y(t) = \int v(t) dt + C \quad (2)$$

↑ explicitly account for arb. constant.

Constant C is found from the initial condition.

However, we now use an integration method which will find C right away. (We'll use the same method many times in this course.)

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$$\int_0^t y'(t_1) dt_1 = \int_0^t v(t_1) dt_1$$

we will explain this notation later

By Fund. Thm. of Calculus = $y(t) - y(0)$.
so:

$$y(t) - y(0) = \int_0^t v(t_1) dt_1$$

$$y(t) = y_0 + \int_0^t v(t_1) dt_1 \quad (3)$$

Moral: Our goal is:

Given the diff. eqn. $\frac{dy}{dt} = f(t, y)$ (4a)

and initial condition $y(t_0) = y_0$ (4b)

We want to find solution $y(t)$.

Note 1 The combination of diff. eqn., (4a), and initial condition, (4b), is called initial value problem (IVP). (DE)

Note 2 When we solve any DE, we get an arbitrary constant, C (see (2)). Its value is found from the initial cond.; e.g., in (3), $C = y_0$.

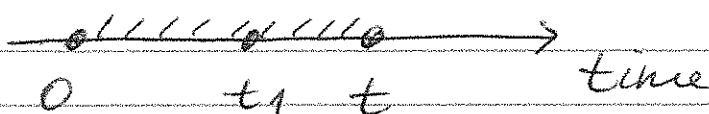
Note 3 (about notations)

(1-3)

When we integrate $v(t)$ from 0 to t , we write:

$$\int_0^t v(t_1) dt_1$$

this is the time up to which we integrate



This is a "dummy" variable $(0 \leq t_1 \leq t)$.

It must have a different name

than the finite integration time.

Sometimes, we may use $\tilde{t}$ (t -tilde) instead of t_1 :

$$\int_0^t v(\tilde{t}) d\tilde{t}$$

Note 4 DE (4a) is more complicated than DE (1). Indeed, if we write its solution as:

$$y(t) = \int f(t, y(t)) dt + C, \quad (5)$$

we see that we cannot integrate because we do not know the unknown solution $y(t)$ inside $f(t, y)$.

So we need to develop methods of how to find $y(t)$ other than by (5).

(2) Examples of DEs

Ex. 2 Given IVP: $y' = a y$ $\swarrow$ const (6)

$$y(0) = 3$$

a) Show that the solution of this DE is

$$y = C e^{at} \quad (C = \text{const})$$

b) Find the solution of the IVP.

Sol'n: a) We are asked to verify a given solution, so we simply compare the lhs. and rhs,

lhs: $y' = (C e^{at})' = C (e^{at})' \xrightarrow{\text{Chain Rule}} C \cdot (at)' \cdot e^{at} = C \cdot a \cdot e^{at}$

rhs: $a \cdot y = a \cdot C e^{at}$

$\text{lhs} = \text{rhs.} \quad \checkmark$

b) $C e^{at} \Big|_{t=0} = 3 \Rightarrow C \cdot \cancel{e^0} = 3 \Rightarrow C = 3,$

$$y(t) = 3 e^{at}.$$

Note: DE (6) is the most basic DE which arises in many applications in all branches of science (physics, chemistry, ...)

It basically says:

$$\underbrace{y'}_{\substack{\uparrow \\ \text{rate of change} \\ \text{of } y}} = \underbrace{a}_{\substack{\uparrow \\ \text{is proportional} \\ \text{to}}} \cdot \underbrace{y}_{\substack{\uparrow \\ \text{instantaneous} \\ \text{value of } y.}}$$

For example:

sub-Ex. 2(a)

$$\frac{dm}{dt} = -a \cdot m$$

mass of radioactive material
(radioactive decay law :
a given percentage of material decays in
a unit of time).

sub-Ex. 2(b) ← Newton's Cooling Law

$$\frac{d(T - T_r)}{dt} = -k \cdot (T - T_r)$$

T  coffee
 T_r = room temp.
↑
temperature of coffee

temp. of coffee
in excess of
room temperature

sub-Ex. 2(c)

$$\frac{dn}{dt} = b \cdot n$$


number of bacteria
multiplying in presence of unlimited
supply of food.

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③ Some terminology

1) $y'(t) = f(t, y(t))$

$t \leftarrow$ independent variable

$y \leftarrow$ dependent variable

2) Ordinary DE : $\frac{dy}{dt}$ is ordinary derivative.

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Partial DE : $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$
involves partial derivatives of dependent variable $u(x,t)$ w.r.t. two independent variables x, t .

3) Order of DE = order of highest derivative

Ex. 3(a)

$$y'' = 1$$

$\uparrow$ 2nd der., $\Rightarrow$ order = 2

Ex. 3(b)

$$\sin(y') + \cos(y''') = 0$$

$\uparrow$ 3rd der., $\Rightarrow$ order = 3

Ex. 3(c)

$$(y'')^5 + (y')^2 + \ln y = 0$$

$\uparrow$ 2nd der., $\Rightarrow$ order = 2.

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4) Autonomous & non-autonomous DE.

$$y' = f(y) \leftarrow \text{autonomous.}$$

$$y' = y^2 - \sin(e^y).$$

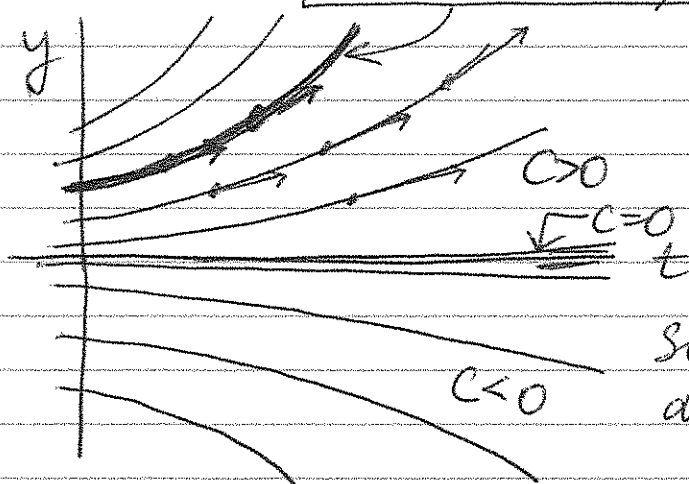
$$y' = f(t, y) \leftarrow \text{non-autonomous}$$

$$y' = y^2 - t$$

④ Direction fields

Idea: Recall Ex. 2.

IVP has 1 solution $y(t)$.



But DE has
 ∞ many solutions
because of
arb. const C .

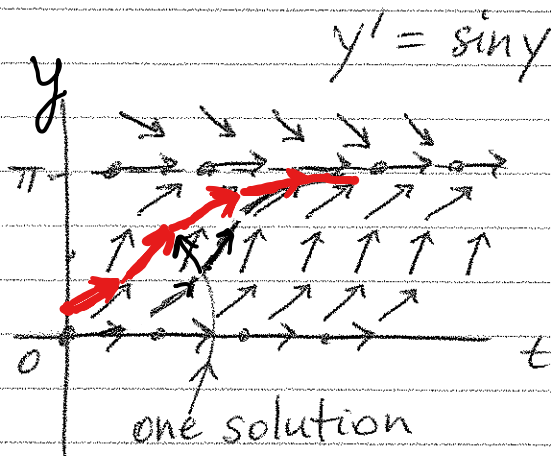
So sol'ns of DE
are many curves.

Direction field is the collection of all
these curves with arrows at each point
indicating the direction of motion.

See also a more detailed version of this example posted next to this Lecture.

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Ex. 4 Sketch the direction field of



Sol'n:

This is an autonomous DE, $\Rightarrow$

y' is the same along each line $y = \text{const.}$

$$y = 0 \Rightarrow \sin y = 0 \Rightarrow y' = 0$$

$$y = \frac{\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \approx 0.7, \quad y' \approx 0.7$$

$$y = \frac{\pi}{2} \Rightarrow \sin y = 1, \quad y' = 1$$

$$y = \frac{3\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \approx 0.7, \quad y' \approx 0.7$$

$$y = \pi \quad \sin y = 0, \quad y' = 0$$

$$y = \frac{5\pi}{4} \quad \sin y = -\frac{1}{\sqrt{2}} \approx -0.7, \quad y' \approx -0.7$$

Equilibrium solution:

A constant solution such that $y' = 0$ for all t :

$$y' = f(t, y_0) = 0.$$

That is, $y(t) = y_0$ is a sol'n for all t .

Ex. 5(a) In Ex. 4, equilibrium sol'ns are $y=0, \pi, 2\pi$, etc.

Ex. 5(b) In $y' = t(y+1)^2$,
the equilibrium sol'n is $y = -1$.

Ex. 5(c) In $y' = ty + 1$,
there are no equilibrium solutions,
because $y' = 0$ implies $y = -1/t$, which
contradicts $y' = 0$ ($(-1/t)' = 1/t^2 \neq 0$).

Ex. 5(d) In $y' = y^2 + 1$,
there are also no equilib. solutions
because $(y^2 + 1) > 0$ for all y .