

Lecture 10, Introduction to 2nd-order DEs: motivation, basic properties, linear DEs.

① Definition

$$y'' = F(t, y, y') \quad (1)$$

is called a 2nd-order DE
(shorthand notation: DE-2, as opposed
to the 1st-order DEs, or DE-1).

If F is a nonlinear function of y
and/or y' , then DE-2 can be solved
analytically (i.e. not numerically by a
computer) only in very few special cases
(even much more rarely than a nonlinear
DE-1). E.g., $y'' = f(y)$ (an autonomous
eq.) cannot be solved except for some
very special $f(y)$.

Then we look at the linear DE-2:

$$\rightarrow y'' + p(t)y' + q(t)y = g(t). \quad (2)$$

Unlike linear DE-1, Eq. (2) cannot be
solved for general $p(t)$ & $q(t)$. This is a
big difference between linear DE-1 & DE-2.
We'll show it later in this lecture.

In general, solving a DE-2 is much
harder than a DE-1.

However, there are also some similarities in
their properties.

We'll study only this DE
in Chap. 3

② Motivation

Q: Why consider DE-2?

A: Most processes in Mechanics are governed by the 2nd Law of Newton:

$$m \vec{a} = \sum \vec{F}$$

acceleration $\uparrow$ $\nwarrow$ vector sum of all forces

In one dimension, $\vec{a} = \frac{d^2 y}{dt^2}$, $\Rightarrow$

$$m y'' = F(t, y, y') \quad (3)$$

position $\uparrow$ $\nwarrow$ velocity

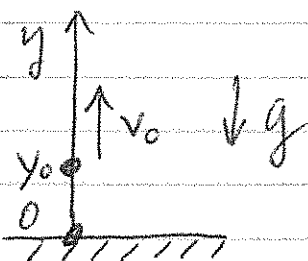
which has the form (1).

③ What to expect of the solution?

Consider a very simple DE-2 (HS Physics or Calc. III):

$$m y'' = -mg \quad \leftarrow \text{gravity}$$

$$\begin{cases} y'' = -g \\ y(0) = y_0, y'(0) = v_0 \end{cases}$$



Solution: 1) $y'' = -g \Rightarrow y' = -\int g dt = -gt + C_1$
 $y'(0) = v_0 = -g \cdot 0 + C_1 \Rightarrow C_1 = v_0.$

$$2) \quad y' = -gt + v_0 \Rightarrow y = \int (-gt + v_0) dt \\ = -gt^2/2 + v_0 t + C_2.$$

$$y(0) = y_0 = -g \cdot 0 + v_0 \cdot 0 + C_2 \Rightarrow C_2 = y_0.$$

$$\text{Thus } y(t) = -\frac{gt^2}{2} + v_0 t + y_0$$

Observations: 1) We have two constants, C_1 & C_2 , not one as for DE-1.

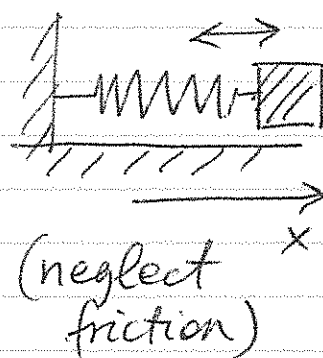
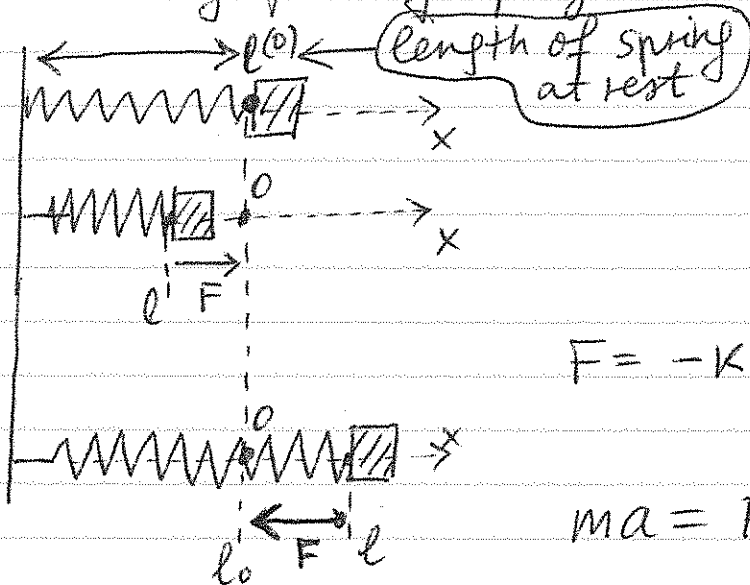
2) We need two initial cond's, $y(0)$ & $y'(0)$, to determine these constants (although, in general, they depend on $y(0)$, $y'(0)$ in a more complicated way).

Meanings: $y(0)$ = initial position; $y'(0)$ = initial velocity.

④ A fundamental model: linear oscillator.

Consider the motion of a mass on a spring:

1) Restoring force of spring



$$F = -k(\underbrace{l - l_0}_x);$$

$$ma = F \Rightarrow$$

(10-4)

$$mX'' = -kX$$

$$mX'' + kX = 0$$

$$X'' + \left(\frac{k}{m}\right)X = 0$$

$\rightarrow \omega^2 \leftarrow$ "omega square"

$$\boxed{X'' + \omega^2 X = 0}$$

$\leftarrow (4)$

Since mass on a spring oscillates,

Eq. (4) is called the

linear oscillator model.

It is fundamental to DE-2s, just like
 $y' = ay$ is fundamental for DE-1.

2) General sol'n (to be derived later, in
Sec. 3.5)

a) $X_1 = \cos \omega t$ is a sol'n of (4).

Check: $X_1' = -\omega \sin \omega t$

$$X_1'' = -\omega \cdot \omega \cdot \cos \omega t = -\omega^2 \cos \omega t.$$

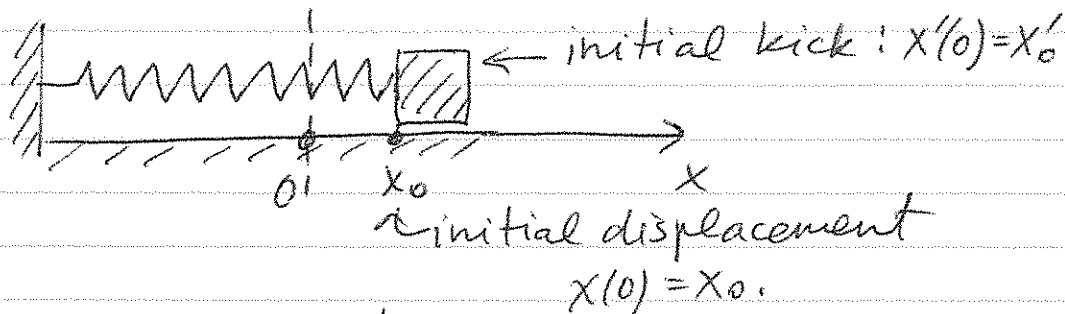
$$X_1'' + \omega^2 X_1 = -\omega^2 \cos \omega t + \omega^2 \cos \omega t = 0 \quad \checkmark.$$

b) $X_2 = \sin \omega t$ is also a sol'n of (4)
(proof: @ home)

c) $X = c_1 X_1 + c_2 X_2 = c_1 \cos \omega t + c_2 \sin \omega t$; $c_{1,2} = \text{const}$
is also a solution of (4) (proof: @ home).

Note: Looks like the superposition principle
for DE-1.

3) Solution for given init. cond's.



Then:

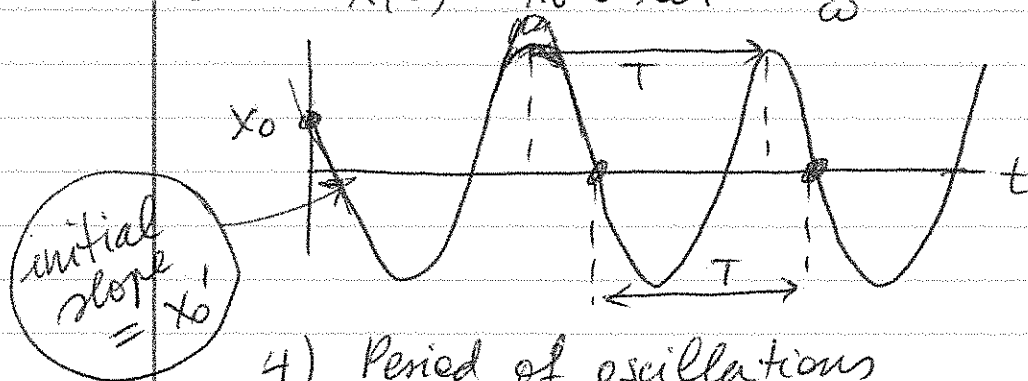
$$x(0) = c_1 \cos(\omega \cdot 0) + c_2 \sin(\omega \cdot 0) = x_0$$

$$x'(0) = c_1 (\cos \omega t)' \Big|_{t=0} + c_2 (\sin \omega t)' \Big|_{t=0}$$

$$= -c_1 \omega \sin(\omega \cdot 0) + c_2 \omega \cos(\omega \cdot 0)$$

$$\Rightarrow \begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= x_0 \\ c_1 \cdot 0 + c_2 \cdot \omega &= x'_0 \end{aligned} \Rightarrow \begin{aligned} c_1 &= x_0 \\ c_2 &= x'_0 / \omega \end{aligned}$$

So: $x(t) = x_0 \cos \omega t + \frac{x'_0}{\omega} \sin \omega t$



Note: $\cos(\omega t_1 + 2\pi) \equiv \cos(\omega[t_1 + \frac{2\pi}{\omega}])$
 $= \cos \omega t_1$ (due to 2π -periodicity)

Similarly, $\sin(\omega t_1 + 2\pi) = \sin \omega t_1$

Thus the solution exactly repeats itself

We will see examples of this later.

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after $t_2 - t_1 = 2\pi/\omega$. So $T = 2\pi/\omega$ is called the period of ^{the} oscillations.

$\omega = \text{frequency},$	$T = \text{period}$
$\omega = \frac{2\pi}{T}$	$T = \frac{2\pi}{\omega}$

(5)

⑤ Existence & uniqueness of solution of a linear DE-2

Thm. 3.1

Let $p(t), q(t), g(t)$ be continuous on $t \in (a, b)$ and let $t_0 \in (a, b)$. Then the sol'n of IVP

$$y'' + p(t)y' + q(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

exists & is unique for $t \in (a, b)$.

Note: This statement is exactly the same as for a DE-1 (Sec. 2.1), except that here we also have an extra coeff. $q(t)$.

⑥ Linear DE-2 cannot be solved analytically for general $p(t)$ & $q(t)$.

Note 1 This is in stark contrast to the situation with linear DE-1, which can be solved for any $p(t), g(t)$ (see Lec. 2).

Note that this Thm is about the solution of the IVP, not of the DE! If the condition of the Thm. is violated, the solutions of the DE may still exist, but we will not be able to solve the IVP.

Note 2 We'll demonstrate our statement for a homogeneous linear DE-2 ($q(t) = 0$).

Later we'll show that if a homogeneous linear DE-2 can be solved, then so can be its non-homogeneous version, too.

We are going to show that a linear DE-2 is equivalent to the Riccati DE-1, which cannot be solved in general (lec. 7).

Consider

$$y'' + p(t)y' + q(t)y = 0 \quad (6)$$

and a change of variables:

$$y = e^{-\frac{\int v(t) dt}{\Gamma(t)}} \quad (7)$$

where $v(t)$ is the new variable.

To substitute (7) into (6), we need y' & y'' .

$$a) \quad y' \stackrel{\uparrow \text{Chain R.}}{=} (-\Gamma)' \cdot e^{-\Gamma} = -v \cdot e^{-\Gamma} \quad \text{product R.}$$

$$b) \quad y'' = (y')' = (-v \cdot e^{-\Gamma})' = -v' \cdot e^{-\Gamma} - v \cdot (e^{-\Gamma})' \\ = -v' \cdot e^{-\Gamma} - v \cdot (-v e^{-\Gamma}) = -v' \cdot e^{-\Gamma} + v^2 e^{-\Gamma}$$

Substitute these into (6):

$$\underbrace{(-v' \cdot e^{-\Gamma} + v^2 e^{-\Gamma})}_{y''} + p(t) \underbrace{(-v \cdot e^{-\Gamma})}_{y'} + q(t) \cdot \underbrace{e^{-\Gamma}}_y = 0$$

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$$-v' + v^2 - p(t)v + q(t) = 0$$

$$v' + p(t)v = v^2 + q(t) \leftarrow \text{Riccati for } v.$$

Since one cannot find v for general p & q ,
so one cannot find y from (7) for
general p & q .

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