

Lecture 12. Homogeneous linear DE-2 with constant coefficients.

We'll consider

$$ay'' + by' + cy = 0, \quad (1)$$

where $a, b, c = \text{const.}$

① The idea behind finding solutions of (1).

Recall that if $\lambda = \text{const.}$, then

$$(e^{\lambda t})' = \lambda e^{\lambda t}; \quad (e^{\lambda t})'' = ((e^{\lambda t})')' = \lambda \cdot \lambda e^{\lambda t} = \lambda^2 e^{\lambda t}.$$

Then seek sol'n of (1) as $y = e^{\lambda t}$:

$$a \cdot (e^{\lambda t})'' + b \cdot (e^{\lambda t})' + c \cdot (e^{\lambda t}) = 0$$

$$a \cdot \lambda^2 e^{\lambda t} + b \cdot \lambda e^{\lambda t} + c \cdot e^{\lambda t} = 0$$

Thus, find λ in $y = e^{\lambda t}$, we need to solve

$$\boxed{a\lambda^2 + b\lambda + c = 0} \quad (2)$$

Characteristic eq. for (1);

$P(\lambda) = a\lambda^2 + b\lambda + c \leftarrow$ characteristic polynomial.

Ex. 1 Consider

$$y'' + 8y' + 15y = 0$$

(a) Solve the characteristic eqn. for it.

$$\lambda^2 + 8\lambda + 15 = 0$$

$$\lambda = \frac{-8 \pm \sqrt{64 - 4 \cdot 15}}{2} = \frac{-8 \pm 2}{2} = -4 \pm 1 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

(b) Show that $y_1 = e^{-5t}$ & $y_2 = e^{-3t}$ form a fundamental set (see Lec. 11) of solutions of the DE. Write the general sol'n.

$$W(t) = \begin{vmatrix} e^{-5t} & e^{-3t} \\ -5e^{-5t} & -3e^{-3t} \end{vmatrix} = -3 \cdot e^{-5t} \cdot e^{-3t} - (-5e^{-5t}) \cdot e^{-3t}$$

$$= (5-3) \cdot e^{-5t-3t} = 2e^{-8t} \neq 0 \text{ for all } t,$$

$\Rightarrow y_1$ & y_2 form a fund. set.

General sol'n: $y = c_1 e^{-5t} + c_2 e^{-3t}$. ✓

② Various cases for roots of the char. eqn.

$$P(\lambda) \equiv a\lambda^2 + b\lambda + c = 0 \quad (2)$$

$$\Rightarrow \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

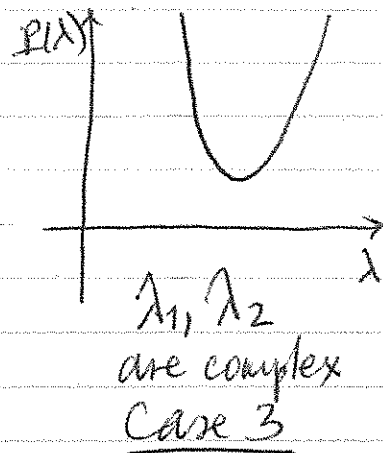
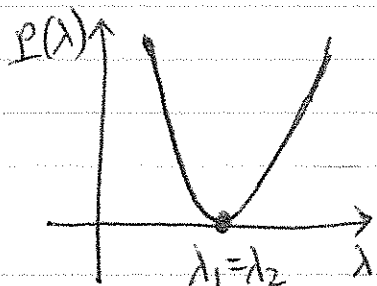
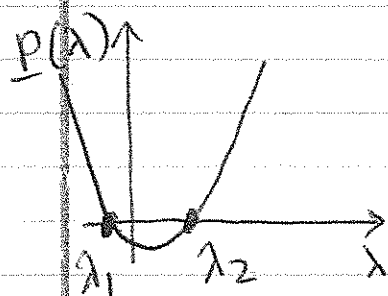
Note: When $a=1$, the following alternative formula may be convenient:

$$\lambda^2 + p\lambda + q = 0 \quad (3)$$

$$\Rightarrow \lambda = -\left(\frac{p}{2}\right) \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

There are 3 cases, depending on the discriminant $(b^2 - 4ac)$ of (2).

Case 1 $b^2 - 4ac > 0$; $\Rightarrow P(\lambda)$ has two real, distinct roots λ_1, λ_2 .



Case 1

Case 2

Case 3

Then $y_1 = e^{\lambda_1 t}$, $y_2 = e^{\lambda_2 t}$ form a fund. set of sol'n's of (1) and $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ is the general sol'n.

See Ex. 1 above & top of p. 124 in book.

Case 2 $b^2 - 4ac = 0$; $\Rightarrow P(\lambda)$ has only one, real root $\lambda_1 (= \lambda_2)$.

Then $y_1 = e^{\lambda_1 t}$, and a second sol'n y_2 , needed for a fundamental set, is missing. We'll show how to find it in Lec. 13.

Note: $y_2 \neq c y_1$: see Ex. 1(b) in Lec. 11 (similar idea, though different functions).

Case 3 $b^2 - 4ac < 0$; $\Rightarrow P(\lambda)$ has two complex sol'n's λ_1, λ_2 :

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$$\lambda_{1,2} = \frac{-b \pm \sqrt{-(4ac-b^2)}}{2a} = \frac{-b \pm i\sqrt{4ac-b^2}}{2a},$$

where $i = \sqrt{-1}$ (or $i^2 = -1$),

A relation of this complex solution to a real-valued sol'n of the IVP will be considered in Lec. 14.

Ex. 2 ($\approx$ Ex. 3 in book); is related to Case 1.

(a) Solve the IVP

$$y'' + y' - 2y = 0, \quad y(0) = y_0, \quad y'(0) = y'_0.$$

Sol'n: 1) Characteristic eq:

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow (\lambda + 2)(\lambda - 1) = 0 \\ \Rightarrow \lambda_1 = -2, \lambda_2 = 1.$$

2) General sol'n:

$$y = c_1 e^{-2t} + c_2 e^t$$

3) IVP:

$$y(0) = c_1 \cdot 1 + c_2 \cdot 1 = y_0 \\ y'(0) = c_1 \cdot (-2) + c_2 \cdot 1 = y'_0$$

In matrix form: $\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$

$\det A \equiv W = 1 \cdot 1 - (-2) \cdot 1 = 3 \neq 0$, $\Rightarrow$ there is a unique sol'n for any y_0, y'_0 .
Finding it (by elimination, e.g.):

$$c_1 = \frac{y_0 - y'_0}{3}, \quad c_2 = \frac{2y_0 + y'_0}{3}.$$

$$\text{So: } y = \frac{y_0 - y_0'}{3} e^{-2t} + \frac{2y_0 + y_0'}{3} e^t.$$

(b) For what values of y_0, y_0' can we have $y(t \rightarrow \infty) \rightarrow 0$? $y(t \rightarrow -\infty) \rightarrow 0$?

Sol'n: Note that:

$$\begin{array}{ll} \underline{t \rightarrow +\infty} & e^{-2t} \rightarrow 0, \quad e^t \rightarrow \infty \\ \underline{t \rightarrow -\infty} & e^{-2t} \rightarrow \infty, \quad e^t \rightarrow 0. \end{array}$$

Therefore, for $y(t \rightarrow +\infty) \rightarrow 0$, we must have

$$c_2 = \frac{2y_0 + y_0'}{3} = 0 \Rightarrow 2y_0 + y_0' = 0.$$

For $y(t \rightarrow -\infty) \rightarrow 0$, we must have

$$c_1 = \frac{y_0 - y_0'}{3} = 0 \Rightarrow y_0 - y_0' = 0.$$

In all other cases, we'll have

$$y(t \rightarrow +\infty) \rightarrow \infty \text{ \& } y(t \rightarrow -\infty) \rightarrow \infty$$

