

Lecture 13 Real repeated roots; Reduction of order of a DE.

In Lecture 12 we showed ^{that} if a DE-2 has distinct real roots $\lambda_1 \neq \lambda_2$ of the characteristic equation, then

$$\{y_1 = e^{\lambda_1 t}, y_2 = e^{\lambda_2 t}\} \text{ form a FS.}$$

Hence we have "enough building blocks" to construct the solution of any IVP.

Q: What if $\lambda_1 = \lambda_2$?
Then we have only one solution.
Does it form a FS?

A: No.
There is another solution that we yet need to find.

① Finding the 2nd solution for the FS of a linear homogeneous DE-2 with constant coefficients.

Let the DE be

$$a y'' + b y' + c y = 0 \quad (1)$$

$a, b, c = \text{const}$; and $a=1$ w/o loss of generality.
If we seek $y = e^{\lambda t}$, $\Rightarrow$

$$\lambda^2 + b \cdot \lambda + c = 0, \quad (2)$$

and we know that this quadratic eqn.

has only one root, say $\lambda = \alpha$.

I.e. (2) is equivalent to

$$(\lambda - \alpha)^2 = 0, \quad (3a)$$

or

$$\lambda^2 - 2\alpha\lambda + \alpha^2 = 0. \quad (3b)$$

Thus, we know that the DE must be:

$$y'' - 2\alpha y' + \alpha^2 y = 0. \quad (4)$$

Trick (similar to variation of parameters;
see Lecture 2, topic ③)

Substitute

into (4):

$$y = e^{\alpha t} \cdot u(t)$$

$$\begin{aligned} (f \cdot g)' &= f'g + f \cdot g' \\ (f \cdot g)'' &= (f'g + fg')' = f''g + 2f'g' + fg'' \end{aligned} \quad (5)$$

$$(e^{\alpha t} u)'' - 2\alpha (e^{\alpha t} u)' + \alpha^2 (e^{\alpha t} u) = 0$$

$$\underbrace{(e^{\alpha t})'' \cdot u + 2(e^{\alpha t})' u' + e^{\alpha t} \cdot u''}_{y''} - 2\alpha \underbrace{(e^{\alpha t})' u + e^{\alpha t} \cdot u'}_{y'} + \alpha^2 \cdot \underbrace{(e^{\alpha t} \cdot u)}_{y} = 0$$

$$\Rightarrow \underbrace{[(e^{\alpha t})'' - 2\alpha \cdot (e^{\alpha t})' + \alpha^2 (e^{\alpha t})] \cdot u}_{=0} +$$

because $y = e^{\alpha t}$ solves (4)

$$+ \underbrace{[2 \cancel{(e^{\alpha t})' u} + e^{\alpha t} \cdot u'' - 2\alpha \cancel{e^{\alpha t} \cdot u'}]}_{=0} = 0$$

$$\Rightarrow e^{\alpha t} \cdot u'' = 0$$

$$\Rightarrow u'' = 0$$

We integrate this twice (lecture 10, topic 3; or Calc. I or Calc. III ← motion with constant acceleration) and find:

$$u = a_1 t + a_2, \quad (6)$$

$$a_1, a_2 = \text{const.}$$

Combining (5) & (6):

$$y_2 = (a_1 t + a_2) e^{\alpha t} \quad (7)$$

is the 2nd solution of DE (1).
Computing Wronskian of $y_1 = e^{\alpha t}$
and y_2 given by (7), one finds

$$W \neq 0 \quad (\text{see p. 128}).$$

Thus, $\{y_1, y_2\}$ form a FS of solutions of (1),
and so the general sol'n is:

$$y = c_1 e^{\alpha t} + c_2 (a_1 t + a_2) e^{\alpha t}$$

$$= \underline{c_1 e^{\alpha t}} + c_2 a_1 t e^{\alpha t} + \underline{(c_2 a_2) e^{\alpha t}}$$

$$= \underline{(c_1 + c_2 a_2) e^{\alpha t}} + \underline{(c_2 a_1) \cdot t e^{\alpha t}}$$

rename: C_1 C_2

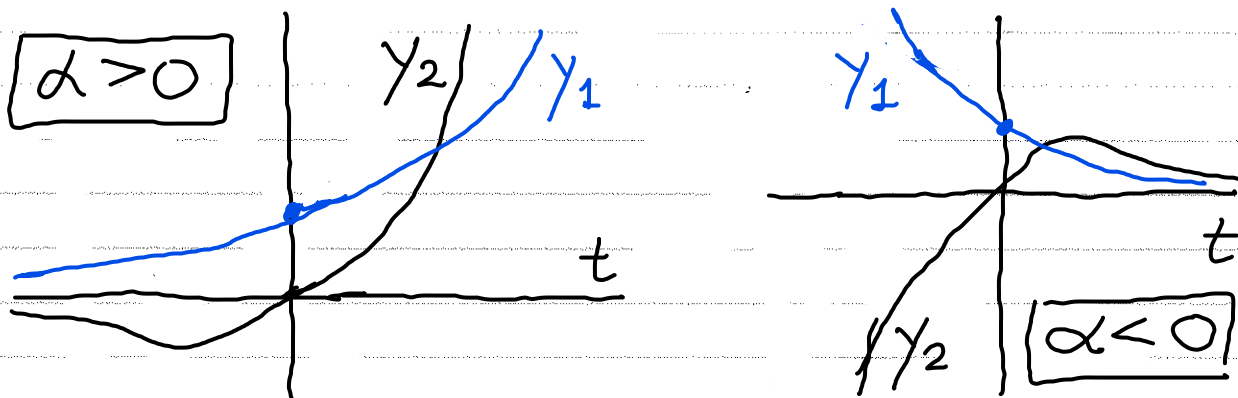
$$= C_1 e^{\alpha t} + C_2 \cdot t e^{\alpha t}.$$

13-4

Thus, we can take as the FS:

$$\boxed{y_1 = e^{\alpha t}, \quad y_2 = te^{\alpha t}} \quad (8)$$

These functions look as follows (MUST MEMORIZE!):



② Finding the second solution of a homogeneous linear DE-2;
The general case of Reduction of Order.

The same trick as above also works for a general homogeneous linear DE-2:

$$y'' + p(t)y' + q(t)y = 0 \quad (9)$$

Suppose we know one its sol'n:

$$y = y_1(t). \quad (10)$$

$$\text{Then seek } y_2 = y_1(t) \cdot u(t). \quad (11)$$

See Ex. 2 + p. 129 in textbook & the following example.

Ex. 1 DE

$$t y'' + (2t-1)y' + (t-1)y = 0 \quad (12)$$

- (a) has a sol'n $y_1 = e^{-t}$, which can be verified by inspection. Find y_2 from (11) and verify that these y_1, y_2 form a fundamental set for some $t \in (a, b)$. Identify this (a, b) .

- (b) Solve the IVP consisting of (12) & IC:

$$y(0) = 0, \quad y'(0) = 1 \quad (13)$$

and explain the result.

Sol'n: (a) 1) $y_2 = e^{-t} \cdot u$
substitute into (12):

$$t [e^{-t} \cdot u'' + 2 \cdot (e^{-t})' \cdot u' + \underline{(e^{-t})'' \cdot u}] +$$

$$(2t-1) \cdot [e^{-t} \cdot u' + \underline{(e^{-t})' \cdot u}] +$$

$$\underline{(t-1) \cdot e^{-t} \cdot u} = 0 \quad \Rightarrow$$

$$\{ t e^{-t} u'' + t \cdot 2 (e^{-t})' u' + (2t-1) \cdot e^{-t} \cdot u' \} +$$

$$u \cdot [t (e^{-t})'' + (2t-1) (e^{-t})' + (t-1) e^{-t}] = 0$$

$= 0$ since e^{-t} is a sol'n of (12)

$\Rightarrow$

13-6

$$e^{-t} t u'' - 2t e^{-t} u' + (2t - 1) e^{-t} u' = 0$$

$\Rightarrow$

$$t u'' - u' = 0 \quad | \quad u' \equiv v$$

$$t v' - v = 0 \quad \Rightarrow$$

$$v = Ct \quad (\text{Lecture 2, Ex. 6})$$

$$\Rightarrow u = \int Ct dt = \frac{C}{2} t^2 + C_0. \quad \text{Thus}$$

$$y_2 = \left(\frac{C}{2} t^2 + C_0 \right) e^{-t}.$$

Note: General sol'n:

$$y = \underline{C_1} e^{-t} + C_2 \left(\frac{C}{2} t^2 e^{-t} + \underline{C_0 e^{-t}} \right)$$

$$= \underbrace{(C_1 + C_2 C_0)}_{C_1} e^{-t} + \underbrace{\left(\frac{C_2 C}{2} \right)}_{C_2} t^2 e^{-t}.$$

Thus, can take $y_1 = e^{-t}$, $y_2 = t^2 e^{-t}$.

$$2) \quad W = \begin{vmatrix} e^{-t} & t^2 e^{-t} \\ -e^{-t} & (2t e^{-t} - t^2 e^{-t}) \end{vmatrix}$$

$$= (2t e^{-2t} - \cancel{t^2 e^{-2t}}) + (\cancel{t^2 e^{-2t}})$$

$$= 2t e^{-2t}.$$

$W \neq 0$ for $t > 0$ or $t < 0$.

This makes sense because (12) can be written as

$$y'' + \underbrace{\left(\frac{2t-1}{t}\right)}_{p(t)} y' + \underbrace{\left(\frac{t-1}{t}\right)}_{q(t)} y = 0,$$

and p, q are continuous for $t > 0$ or $t < 0$.

Thus, $\{e^{-t}, t^2 e^{-t}\}$ is a FS for $t \in (0, \infty)$ or $t \in (-\infty, 0)$.

$$(b) \quad y(0) = 0 \Rightarrow (c_1 e^{-t} + c_2 t^2 e^{-t}) \Big|_{t=0} = 0$$

$$c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y'(0) = 1 \Rightarrow (c_1 (-e^{-t}) + c_2 (2t - t^2) e^{-t}) \Big|_{t=0} = 0$$

$$-c_1 + c_2 (0 - 0) = 1.$$

$$\text{Thus } \begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \Rightarrow \begin{cases} c_1 + 0 \cdot c_2 = 0 \\ -c_1 + 0 \cdot c_2 = 1 \end{cases} \Rightarrow \text{no solutions!}$$

Why? Because our IC is at $t=0$, but $\{y_1, y_2\}$ is not a FS on $[0, \infty)$ because $W(0) = 0$!

Thus, to be able to solve a IVP, we must take $t_0 \neq 0$.