Lecture 14. Complex roots; oscillatory solutions of DE-2

1 Meaning of the complex exponential.

$\boxed{1a}$ In Case 3 listed in Lecture 12,

$$\lambda_{1,2} = \frac{-b}{2a} \pm i\,\frac{\sqrt{4ac-b^2}}{2a}$$

$$\equiv \alpha \pm i\beta, \qquad (1)$$

if $(4ac - b^2 > 0)$ in the characteristic eq.

$$a\lambda^2 + b\lambda + c = 0 \qquad (2)$$

Then the DE-2:

$$ay'' + by' + cy = 0 \qquad (3)$$

has two solutions

$$y_1 = e^{(\alpha+i\beta)t}, \quad y_2 = e^{(\alpha-i\beta)t}. \qquad (4)$$

Moreover, since $\lambda_1 \neq \lambda_2$, then as shown in Lecture 12 and p. 124 of book,

$\{y_1, y_2\}$ is a FS.

3/4/16

Ex. 1 Suppose $a=1$, $b=0$, $c=\omega^2$.

Then (2) becomes

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow$$

$$\lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega. \qquad (5)$$

Then the corresponding DE

$$y'' + \omega^2 y = 0 \qquad (6)$$

has solutions

$$y_1 = e^{i\omega t}, \quad y_2 = e^{-i\omega t}, \qquad (7)$$

and they form a FS.

Q1: From Lec. 10 we know that (6) has a FS $\{\cos\omega t, \sin\omega t\}$. So, how are the complex sol's (7) related to the real sol's $\{\cos\omega t, \sin\omega t\}$?

Q2: In general, what does a complex solution $e^{(\alpha+i\beta)t}$ have to do with real solutions of a DE?

$\boxed{1b}$ Definition of the complex exponential.

In Calculus II, you learned the Maclaurin (or Taylor) series for e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (8)$$

It was proved in Calc. II that this series converges for all real x.

In a higher-level course (Complex Analysis) one proves that it actually converges for all complex x, too, i.e. when

$$x = a + ib.$$

Moreover, in Complex Analysis one also shows that for any x_1 & x_2, one has the familiar rule:

$$\boxed{e^{x_1+x_2} = e^{x_1} \cdot e^{x_2}} \qquad (9)$$

even when x_1, x_2 are complex.

Therefore,

$$e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} \qquad (10)$$

Since we know how to compute $e^{\alpha t}$, we just need $e^{i\beta t}$.

1c Euler's formula

We first create a multiplication table for i:

n	0	1	2	3	4	5	6	
i^n	1	i	-1	$-i$	1	i	-1	...... repeats!
			$(i)^2=-1$	$(i)^2 \cdot i$	$i^4 = i^2 \cdot i^2$	$i^4 \cdot i$	$i^5 \cdot i$	

Now use (8):

$$e^{i\beta t} = 1 + \frac{i\beta t}{1!} + \frac{(i\beta t)^2}{2!} + \frac{(i\beta t)^3}{3!} + \frac{(i\beta t)^4}{4!} + \frac{(i\beta t)^5}{5!} + \frac{(i\beta t)^6}{6!} + \ldots$$

use Table for i^n

$$\overset{\curvearrowright}{=} 1 + i\cdot\frac{\beta t}{1!} - \frac{(\beta t)^2}{2!} - i\,\frac{(\beta t)^3}{3!} + \frac{(\beta t)^4}{4!} + i\,\frac{(\beta t)^5}{5!} - \frac{(\beta t)^6}{6!} + \ldots$$

$$= \Big[\underbrace{1 - \frac{(\beta t)^2}{2!} + \frac{(\beta t)^4}{4!} - \frac{(\beta t)^6}{6!} + \ldots}_{\cos\beta t}\Big] + i\cdot\Big[\underbrace{\beta t - \frac{(\beta t)^3}{3!} + \frac{(\beta t)^5}{5!} + \ldots}_{\sin\beta t}\Big]$$

$$= \cos\beta t + i\cdot\sin\beta t.$$

So:

$$\boxed{e^{i\beta t} = \cos\beta t + i\sin\beta t} \qquad (11)$$

Euler's formula

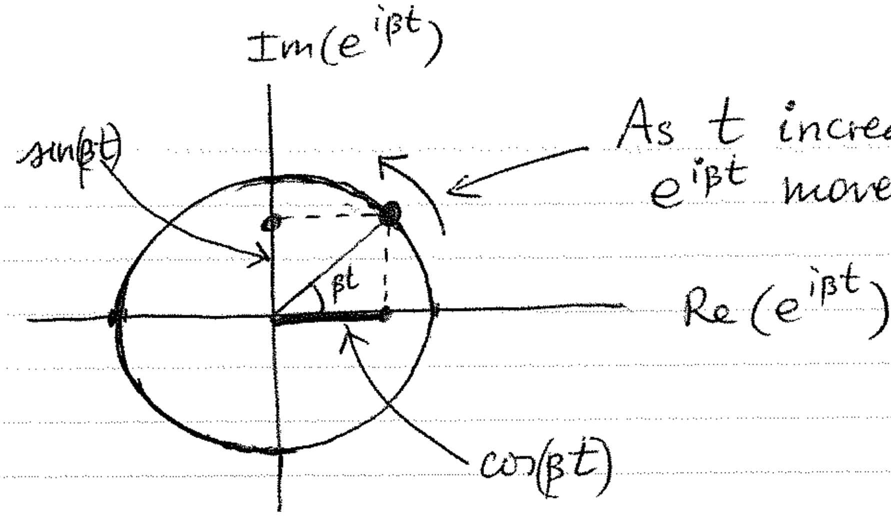
Now: $e^{-i\beta t} = e^{i(-\beta t)} = \cos(-\beta t) + i\cdot\sin(-\beta t)$
$= \cos\beta t - i\cdot\sin\beta t$

$$\boxed{e^{-i\beta t} = \cos\beta t - i\sin\beta t} \qquad (12)$$

Ex. 2 Unit circle and Euler's formula

$$e^{i\beta t} = \underbrace{\cos\beta t}_{\substack{\text{real part} \\ Re(e^{i\beta t})}} + i\cdot\underbrace{\sin\beta t}_{\substack{\text{imaginary part} \\ Im(e^{i\beta t})}}$$

$Re(a+ib) \equiv a;\quad Im(a+ib) \equiv b.$



As t increases, "point" $e^{i\beta t}$ moves along the <u>unit circle</u>!

(2) Constructing a real FS from complex exponentials

$\boxed{2a}$ Method 1

$$y_1 = e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t) \quad (13a)$$

$$y_2 = e^{\alpha t - i\beta t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t}(\cos\beta t - i\sin\beta t) \quad (13b)$$

Since $\{y_1, y_2\}$ is a FS, then

$z_1 = \dfrac{y_1 + y_2}{2}$, $z_2 = \dfrac{y_1 - y_2}{2i}$ are also sol'ns of DE (3) (because any $c_1 y_1 + c_2 y_2$ is a solution).

Thus: $$\boxed{z_1 = e^{\alpha t} \cdot \cos\beta t \; ; \quad z_2 = e^{\alpha t} \cdot \sin\beta t} \quad (14)$$

is another pair of solutions of (3).

Q: Do they form a FS?

A: Yes. See top of p. 136.

3/14/16

$\boxed{26}$ Method 2

Thm. 3.3 (simplified version for DE-2 with constant coefficients; see book for general case).

Consider DE (3):

$$ay'' + by' + cy = 0 \qquad (3)$$

with a, b, c = real constants.
Let

$$y = e^{\alpha t + i\beta t} = e^{\alpha t}(\cos\beta t + i \cdot \sin\beta t)$$
$$= \underbrace{e^{\alpha t}\cos\beta t}_{u(t)} + i \cdot \underbrace{e^{\alpha t}\sin\beta t}_{v(t)}$$

be a sol'n of (3). Then $u(t)$ and $v(t)$ individually are also sol'ns of (3).

Proof: $\quad ay'' + by' + cy = 0$

$$a(u + iv)'' + b(u + iv)' + c \cdot (u + iv) = 0$$

$$\underbrace{(au'' + bu' + cu)}_{\text{real}} + \underbrace{i(av'' + bv' + cv)}_{\text{purely imaginary}} = 0$$

This can hold only when
"real $= 0$" and "imaginary $= 0$"

$\Rightarrow \quad au'' + bu' + cu = 0$ and

$$av'' + bv' + cv = 0.$$

Thus, we've shown that

$$u = e^{\alpha t}\cdot\cos\beta t\,,\quad v = e^{\alpha t}\cdot\sin\beta t$$

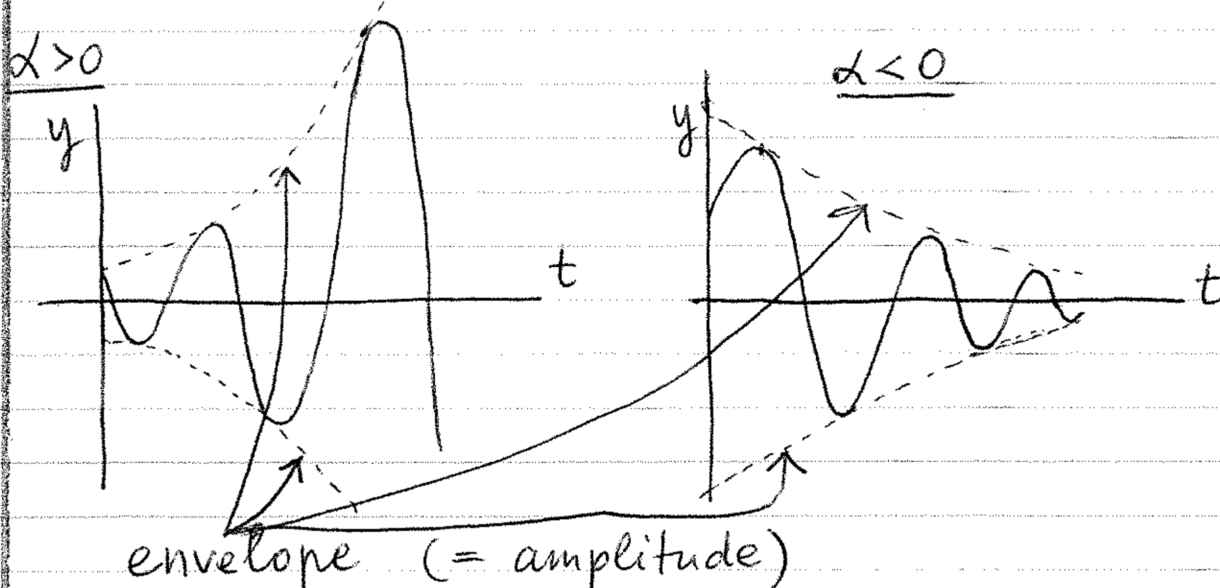
are sol'ns of (3).

2c The general solution

$$\boxed{y = c_1 \overset{u}{z_1} + c_2 \overset{v}{z_2} = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t \equiv A e^{\alpha t}\cos\beta t + B e^{\alpha t}\sin\beta t.} \qquad (15)$$

See Ex. 2 in book for using this general sol'n to solve an IVP.

Note: Let us plot a representative solution (for some A & B)



3 Amplitude and phase of solution

Trig. identity:

$$\boxed{\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b}$$

note the sign!

Sanity check: $a = b$

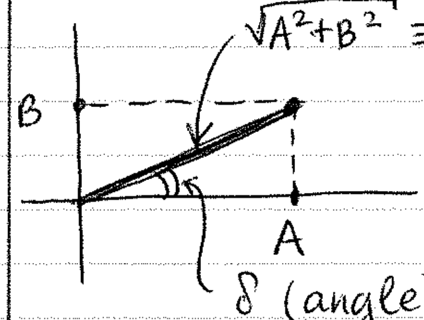
$$\underbrace{\cos(a-a)}_{1} = \underbrace{\cos a \cdot \cos a}_{\cos^2 a} + \underbrace{\sin a \cdot \sin a}_{\sin^2 a}$$

$$1 = \cos^2 a + \sin^2 a \quad \checkmark$$

Now let us "walk backwards" from r.h.s. to l.h.s.

$$A\cos\beta t + B\sin\beta t = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\beta t + \frac{B}{\sqrt{A^2+B^2}}\sin\beta t\right)$$

$\sqrt{A^2+B^2} \equiv R$ (radius in polar coord's)



$$\cos\delta = \frac{A}{\sqrt{A^2+B^2}}$$

$$\sin\delta = \frac{B}{\sqrt{A^2+B^2}}$$

$$\boxed{\tan\delta = \frac{B}{A}}$$

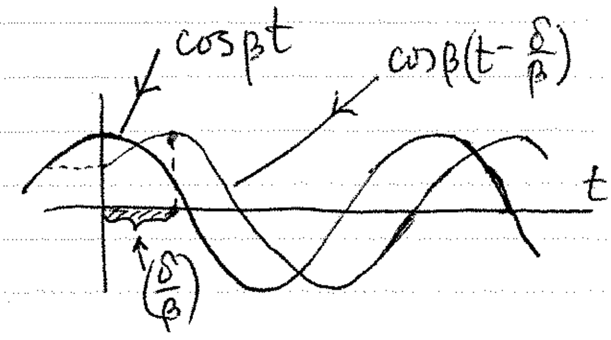
Then

$$\frac{A}{\sqrt{A^2+B^2}}\cos\beta t + \frac{B}{\sqrt{A^2+B^2}}\sin\beta t = \cos\delta\cos\beta t + \sin\delta\sin\beta t$$

$$= \cos(\beta t - \delta).$$

Meaning of "$-\delta$"

$$\cos(\beta t - \delta) = \cos\left(\beta\left[t - \frac{\delta}{\beta}\right]\right)$$



So $\left(\frac{\delta}{\beta}\right)$ represents the "*phase shift*" of the cosine function.

> Note: the book incorrectly refers to δ as the phase!

Now look at the general sol'n of DE (3):

$$y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$$

$$= e^{\alpha t}\cdot\sqrt{A^2+B^2}\cdot\left(\frac{A}{\sqrt{A^2+B^2}}\cos\beta t + \frac{B}{\sqrt{A^2+B^2}}\sin\beta t\right)$$

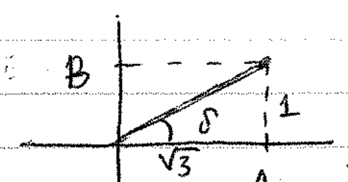
$$= e^{\alpha t}\cdot R\cos(\beta t-\delta) = e^{\alpha t}\cdot R\cdot\cos\beta\left(t-\frac{\delta}{\beta}\right).$$

<u>Ex. 3(a)</u> Rewrite

$$y = \sqrt{3}\cdot\cos\frac{t}{2} + \sin\frac{t}{2}$$

in the form $R\cos(\beta t-\delta)$ and sketch it.

<u>Sol'n:</u>

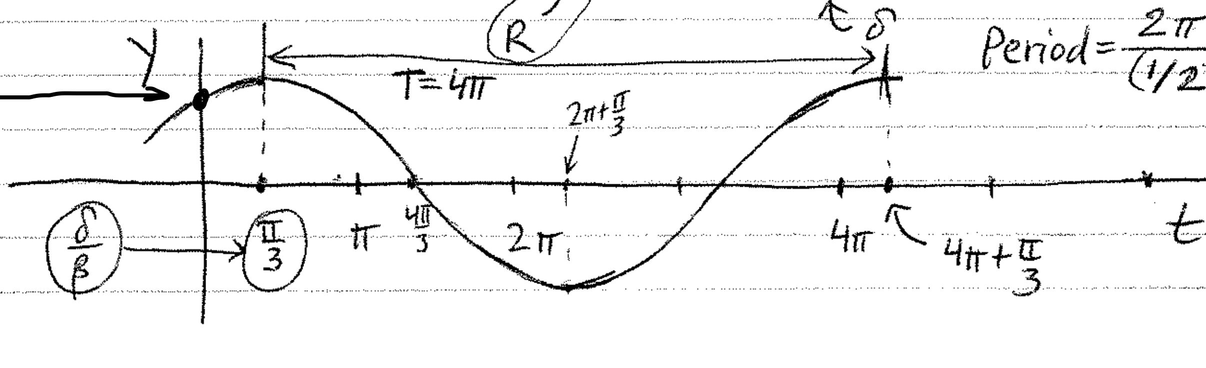


$$\delta = \arctan\frac{B}{A} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

$$R = \sqrt{A^2+B^2} = \sqrt{3+1} = 2.$$

$$y = \underbrace{2}_{R}\cos\left(\frac{t}{2} - \underbrace{\frac{\pi}{6}}_{\delta}\right) = 2\cos\left[\underbrace{\frac{1}{2}}_{\beta}\left(t-\frac{\pi}{3}\right)\right].$$

$$\text{Period} = \frac{2\pi}{(1/2)}$$

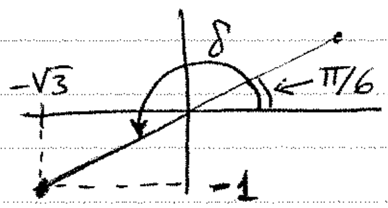


Check: Must agree with $y(0) = \sqrt{3}\cdot\cos 0 + \sin 0 = \sqrt{3}.$ ✓

Ex. 3(b) Same for

$$y = -\sqrt{3}\cos\frac{t}{2} - \sin\frac{t}{2}.$$

Sol'n:



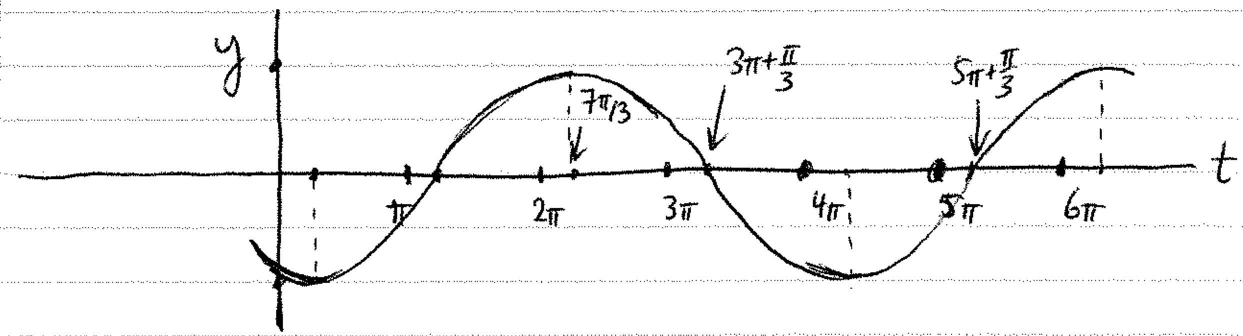
$$\delta = \arctan\left(\frac{-1}{-\sqrt{3}}\right) \underbrace{+\,\pi}$$

$$= \frac{\pi}{6} + \pi,$$

need to add when $A<0$

$$y = 2\cos\left(\tfrac{1}{2}t - \left(\tfrac{\pi}{6}+\pi\right)\right) = 2\cos\left[\tfrac{1}{2}\left(t - \tfrac{7\pi}{3}\right)\right]$$

as in 3(a)



Not surprisingly, this looks like the graph in Ex. 3(a), but flipped about the x-axis.