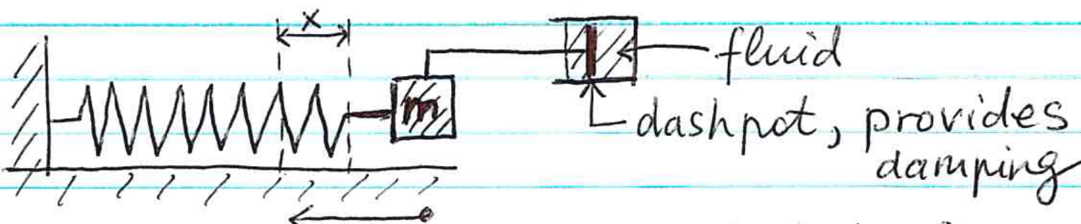
Lecture 15. Unforced mechanical vibrations: linear oscillator model revisited

In Lecture 10 we considered the model of a mass on a spring. Here we will revisit its solution in light of the new mathematical technique learned in Lecture 14, as well as consider its more complicated version, when damping is present.

(1) Viscous damping

1a Horizontal spring (no gravity)



$F_{spring} = -kx$ (Hooke's Law)

$\xleftarrow{\;v\;}$ ← velocity

$$\boxed{F_{Damping} = -\gamma v = -\gamma \frac{dx}{dt}}$$

"gamma"

Newton's 2nd Law of motion:

$$ma = F_{spring} + F_{Damping}$$

$$m\frac{d^2x}{dt^2} = -kx - \gamma\frac{dx}{dt}$$

$$(1) \quad \boxed{mx'' + \gamma x' + kx = 0}$$

linear oscillator with damping

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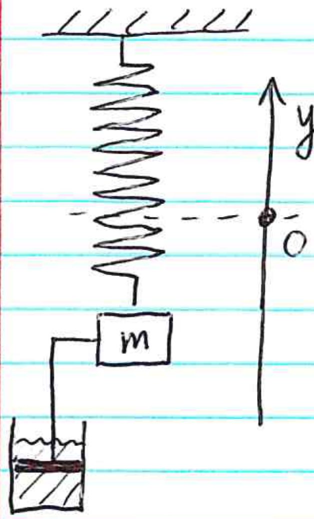
1b Vertical spring

A more realistic situation is when the spring hangs from a ceiling or hook; (this was in HW 10).



$$ma = F_{spring} + F_{damping} + F_{gravity}$$

$$m\frac{d^2y}{dt^2} = -ky - \gamma y' - mg$$

$$\boxed{my'' + \gamma y' + ky = -mg} \quad (2)$$

In equilibrium,

$y' = 0$ (velocity $= 0$)

$y'' = 0$ (acceleration $= 0$)

$\Rightarrow \quad ky_e = -mg \quad \Rightarrow \quad \boxed{y_e = \frac{-mg}{k}} \quad (3)$

(note that $y_e < 0$: see picture)

equilibrium stretch of spring

Let us now show that Eq. (2) can be transformed to have the form of Eq. (1). That is, we need to get rid of "mg" on the r.h.s.

Note that from (3):

$$-mg = ky_e. \quad (4)$$

Then

$$my'' + \gamma y' + ky = k y_e$$

$$my'' + \gamma y' + k(y - y_e) = 0 \quad \Rightarrow$$

since $y_e = \text{const}$

$$m(y-y_e)'' + \gamma(y-y_e)' + k(y-y_e) = 0$$

Same trick as in topic (5) of Lec. 2; see also the Word Problem 2 for HW 10.

If we denote $\boxed{\tilde{y} \equiv y - y_e}$, $\Rightarrow$ the last eqn. becomes:

$$\boxed{m\tilde{y}'' + \gamma \tilde{y}' + k\tilde{y} = 0}$$

This is the same eqn. as (1). ✓

(2) Behavior of the model: 3 cases

The characteristic eqn. for DE (1) is:

$$m\lambda^2 + \gamma\lambda + k = 0 \quad \Rightarrow$$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{(2m)^2} - \frac{\cancel{4}mk}{\cancel{4}m^2}}$$

$$= -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}$$

As in Lec. 10, denote: $\boxed{\frac{k}{m} = \omega^2}$ (5)

ω = angular frequency of undamped oscillations

$\boxed{T = \frac{2\pi}{\omega}}$ = period of undamped oscillations

$f = \frac{1}{T} = \frac{\omega}{2\pi}$ = frequency of undamped oscillations (measured in Hertz; e.g. $f = 100$ Hz $\Leftrightarrow$ $T = 0.01$ sec.)

Also denote $\boxed{\frac{\gamma}{2m} = \alpha}$ (6) ← Note: the textbook has opposite sign.

Then $\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$.

Case 1 : $\boxed{\alpha > \omega}$ $\Rightarrow$ $\lambda_{1,2}$ are real, $\lambda_1 \neq \lambda_2$.

Then the solution of (1), where we replace $x \to y$ to stay with our old notations:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \qquad (7)$$

Since the real exponentials do not oscillate, then the motion is not oscillatory.
Physically, when damping is too high ($\alpha > \omega$), the spring does not vibrate but relaxes to its equilibrium.

This is the overdamped case.

Case 2: $\boxed{\alpha = \omega}$ $\Rightarrow$ $\lambda_1 = \lambda_2$ is real

Then by Lec. 13,

$$y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t}. \quad (8)$$

The damping is still too strong, and there are no vibrations (but 1 zero-crossing can still occur).

This is the critically damped case.

Case 3: $\boxed{\alpha < \omega}$ $\Rightarrow$ $\lambda_{1,2}$ are complex:

$$\lambda_{1,2} = -\alpha \pm i\sqrt{\omega^2 - \alpha^2}$$

$$\equiv -\alpha \pm i\beta,$$

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$$\boxed{\beta = \sqrt{\omega^2 - \alpha^2}} \quad (9)$$

β = angular frequency of damped oscillations

$\boxed{T = \frac{2\pi}{\beta} = \text{period of damped oscillations}}$

$$y(t) = e^{-\alpha t}(c_1 \cdot \cos\beta t + c_2 \cdot \sin\beta t) \quad (10)$$

(from Lecture 14).

Note that damping decreases frequency of oscillations and therefore increases their period.

3 Examples of IVPs

Ex. 1 Undamped oscillations

(a) Find the solution of the IVP

$$my'' + ky = 0$$

$$y(0) = y_0, \quad y'(0) = y_0'$$

Sol'n: 1) General sol'n:

$$y = c_1 \cos\omega t + c_2 \sin\omega t,$$

$$\omega = \sqrt{\frac{k}{m}}$$

2) Initial cond's:

$$y(0) = y_0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = y_0$$

$$y'(0) = y_0' \Rightarrow c_1 \cdot \omega \cdot 0 + c_2 \cdot \omega \cdot 1 = y_0'$$

$$\Rightarrow y = y_0 \cos\omega t + \frac{y_0'}{\omega} \sin\omega t.$$

(b) Find the minimum and maximum values of y.

Sol'n: This requires to cast the sol'n in form:

$$y = R\cos(\omega t - \delta) \leftarrow \text{see Lec. 14.}$$

Repeating from Lec. 14:

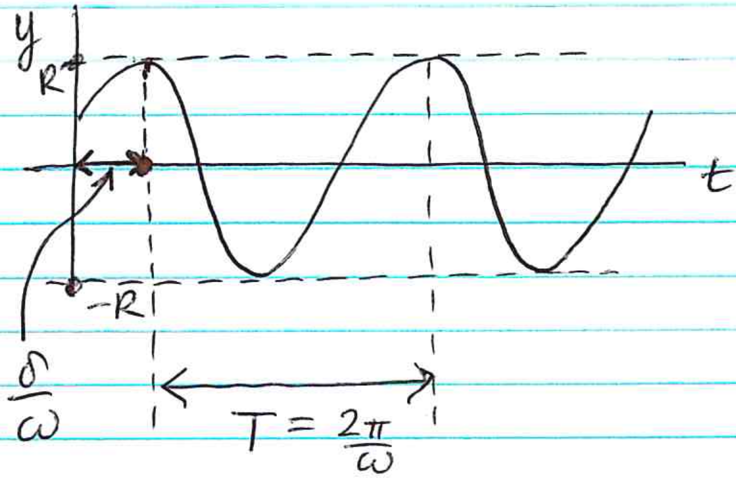
$$c_1 \cos\omega t + c_2 \sin\omega t = \overbrace{\sqrt{c_1^2+c_2^2}}^{R}\Big(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{\cos\delta} \cos\omega t + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{\sin\delta} \sin\omega t\Big)$$

$$= R(\cos\delta \cos\omega t + \sin\delta \cdot \sin\omega t) =$$

$$= R\cos(\omega t - \delta) \equiv R\cdot\cos\left[\omega\left(t - \frac{\delta}{\omega}\right)\right],$$

$$\tan\delta = \frac{c_2/R}{c_1/R} = \frac{c_2}{c_1} = \frac{y_0'}{\omega y_0}$$

$$R = \sqrt{y_0^2 + \left(\frac{y_0'}{\omega}\right)^2}.$$



$y_{max} = R$

$y_{min} = -R$

Ex. 2 Damped oscillations.

Solve the IVP

$$my'' + \gamma y' + ky = 0$$

simplification $y(0) = 0, \quad y'(0) = y_0'$

and sketch the solution.

Sol'n: Using the notations of topic ②:

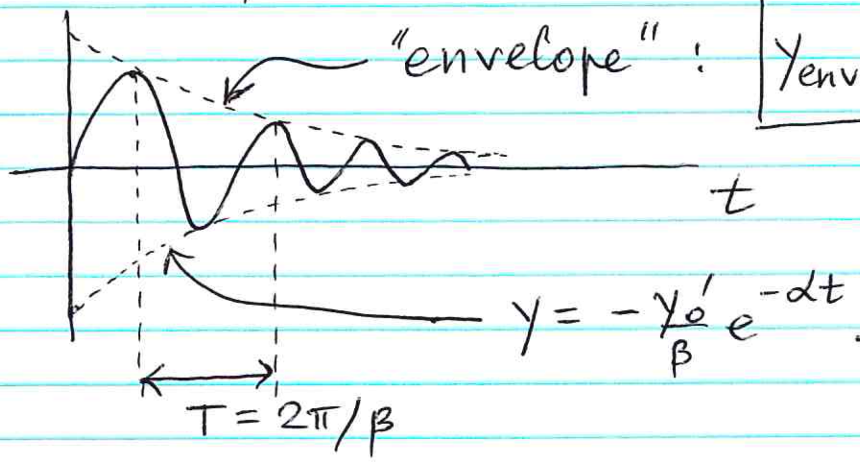
$$y = e^{-\alpha t}(c_1 \cdot \cos\beta t + c_2 \cdot \sin\beta t)$$

IC: $y(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$y'(0) = y_0' \Rightarrow -\alpha \cdot 1 \cdot c_2 \cdot 0 + 1 \cdot c_2 \cdot \beta \cdot 1 = y_0'$

$\Rightarrow \ c_2 = y_0'/\beta$. Thus,

$$y = \frac{y_0'}{\beta} e^{-\alpha t} \sin \beta t .$$



$$\boxed{y_{envelope} = \frac{y_0'}{\beta} e^{-\alpha t}.}$$

Note: Such a simple form occurs only for $y_0 = 0$.