

Lecture 16. General sol'n of linear nonhomogeneous DE.

Here we will obtain the counterpart of the Superposition Principle (part (b)) for linear 2nd-order DEs.

$$y'' + p(t)y' + q(t)y = g(t) \quad (1)$$

① The General sol'n of a NH DE-2.

Q: Suppose $u(t)$ & $v(t)$ are two sol'ns of (1). How different can they be? I.e., what can one say about $(u(t) - v(t))$?

A: $u(t) - v(t) = y_c(t)$,
where $y_c(t)$ is a sol'n of the homogeneous version of (1).

Proof: Since u, v solve (1) $\Rightarrow$

$$u'' + pu' + qu = g$$

$$- \quad v'' + pv' + qv = g$$

$$(u-v)'' + p(u-v)' + q(u-v) = g - g \rightarrow 0,$$

$\Rightarrow (u-v)$ satisfies the homogeneous version of DE. ✓

Notations:

• A solution of the NH DE (1) is called particular sol'n of NH DE: y_p .

- The general sol'n of the homogeneous version of (1) is called the complementary sol'n: y_c .

Then the above result implies:

$$\left(\text{The general sol'n of a NH DE} \right) = \left(\text{A particular sol'n of NH DE} \right) + \left(\text{The general sol'n of the homogeneous DE} \right)$$

$$y(t) = y_p + \underbrace{(c_1 y_1(t) + c_2 y_2(t))}_{y_c} \quad (2)$$

Here $\{y_1, y_2\}$ is a FS of sol'ns of the homogeneous DE.

Note: In Lecture 3 we had:

If y_h is a solution of the hom. DE-1;
and if y_{nh} is a sol'n of the non-hom. DE-1,
then $y = c y_h + y_{nh}$

is a sol'n of the non-hom. DE-1.

So, Eq. (2) is just a generalization of that result for a DE-2.

See Ex. 1 in textbook.

② Superposition Principle for particular sol'ns.

Similarly to Note 1 on p. 3-6 (Lec. 3), if y_{p1} and y_{p2} are two particular sol'ns of the same NH DE, then $(y_{p1} + y_{p2})$ is not a sol'n of that DE. (Proof: WP2 for HW#3.)

However, the following useful modification of the superposition principle for NH DE does hold:

Thm. 3.4 Let u_1 be a sol'n of

$$y'' + p(t)y' + q(t)y = g_1(t);$$

let u_2 be a sol'n of

$$y'' + p(t)y' + q(t)y = g_2(t);$$

let p, q, g_1, g_2 be continuous on $t \in (a, b)$; and let A_1, A_2 be any constants.
Then

$$y_p = A_1 u_1 + A_2 u_2 \quad (3)$$

is a particular sol'n of

$$y'' + p(t)y' + q(t)y = A_1 g_1(t) + A_2 g_2(t). \quad (4)$$

Proof: Similar to that on p. 16-1; see a HW problem.

See Ex. 2 in textbook.

Note 1: The (obvious) significance of Thm. 3.4 is that it allows one to find a particular sol'n of a NH DE whose r.h.s. consists of several terms.

You simply find (see Secs. 3.8 & 3.9) particular sol'ns corresponding to individual terms and then "put them together" as "building blocks".

Note 2 : Recall (similarly to a statement in Lecture 3) that Superposition Principles are valid only for linear DEs.

Neither of Superposition Principles of this Lecture is valid for a nonlinear DE.