

Lec 2. Linear DEs

① Linear & nonlinear DEs. Linear homogeneous & non-homogeneous DEs.

Def: DE $y' = f(t, y)$

is called linear if $f(t, y) = \underbrace{a(t)}_{\text{arbitrary functions of } t} \underbrace{y}_{\text{just } y \text{ (i.e., no } y^2, 1/y, \sin y, \text{ etc.)}} + \underbrace{b(t)}$

Alternative form used
more frequently in this course:

$$y' + p(t)y = g(t) \quad (1)$$

Def: If in (1), $g(t) = 0$ for all t ,
the DE is called homogeneous.
If $g(t) \neq 0$ for some t , $\Rightarrow$ non-homogeneous.

Ex. 1 Classify equations as linear or nonlinear. If they are linear, further classify as homogeneous or non-homogeneous.

(a) $y' = t y^2 \leftarrow$ nonlinear

(b) $y' = (-t^2) y \leftarrow$ linear ($p = -t^2$, $g = 0$)
homogeneous

$$(c) (y')^2 = ty \Rightarrow$$

$$y' = \pm \sqrt{ty} \leftarrow \text{nonlinear}$$

$$(d) (\cos t) y' + e^t \cdot y = \sin t \Rightarrow$$

$$y' + \underbrace{\frac{e^t}{\cos t}}_{p(t)} y = \underbrace{\tan t}_{g(t) \neq 0} \leftarrow \text{linear}$$

$\nwarrow$ non-homogeneous

② Solution of linear homogeneous DE.

$$y' + p(t)y = 0, \text{ or}$$

$$y' = -p(t)y. \quad (2)$$

$$\text{Let } P(t) = \int p(t) dt, \text{ or}$$

$$P'(t) = p(t). \quad (3)$$

The solution of (2) is based on two observations.

$$1) (e^{-P(t)})' \stackrel{\text{Chain Rule}}{=} (-P(t))' \cdot e^{-P(t)} \stackrel{(3)}{=} -p(t) e^{-P(t)}.$$

So denote $z(t) = e^{-P(t)}$; then:

$$z' = -p(t)z. \leftarrow \text{Looks like (2).}$$

So we've found one sol'n of (2).

2) To find the general sol'n, note that

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$y = C \cdot z$, where $C = \text{any const}$,
also satisfies (2):

$$C \cdot (z' = -p z)$$

$$C z' = -C p z$$

$$(Cz)' = -p(Cz).$$

Thus,

$$\boxed{y = C e^{-P(t)}} \quad (4)$$

is the general sol'n of (2).

Note: Since we've, basically, guessed the sol'n $z = e^{-P(t)}$, we cannot yet guarantee that we haven't missed some other sol'ns.

We will prove this ~~completeness~~ in lec. 5.
So for now we take on faith that (4) is the most general sol'n of (2).

③ Sol'n of linear nonhomogeneous DE.

Method 1 Integrating factor

1) Observation about the homogeneous DE (2):
 $e^P \cdot (y' + p y = 0)$

$$e^P \cdot y' + \underbrace{(e^P \cdot p)}_{(e^P)'} \cdot y = 0$$

Product Rule: $(e^P \cdot y)' = 0$

$$e^P \cdot y = C$$

$$y = C e^{-P(t)}. \text{ (Agrees with 4.)}$$

2) Apply this idea to the nonhomogeneous DE:

$$y' + p y = g \quad (5)$$

$$e^P \cdot (y' + p y = g)$$

same as above $\rightarrow (e^P y)' = e^P g$

$$e^P y = \int e^{P(t)} g(t) dt$$

$$y = e^{-P(t)} \int e^{P(t)} g(t) dt \quad (6)$$

is the solution of (5).

Note 1 The indefinite integral in (6) contains, by definition, an arbitrary constant.

Let us explicitly account for this by rewriting:

$$y = e^{-P(t)} \left(\int e^{P(t)} g(t) dt + C \right) \quad (7)$$

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some particular
value of indefinite $\int$

e.g. $\int (t-1) dt = \frac{(t-1)^2}{2}$

Note 2 It may look like $e^{-P(t)}$ outside the $\int$ and $e^{P(t)}$ inside the $\int$ will cancel. However, this is not so.

Ex. 2 let $p(t) = -\frac{2t}{1+t^2}$, $g(t) = 2t$.

Let us set $C=0$ in (7) for now.

$$P(t) = \int p(t) dt = -\int \frac{2t}{1+t^2} dt \Big|_{u=1+t^2} = -\ln(1+t^2).$$

$$e^{P(t)} = e^{-\ln(1+t^2)} = (e^{\ln(1+t^2)})^{-1} = \frac{1}{1+t^2}.$$

$$e^{-P(t)} = (1+t^2).$$

$$y = \underbrace{(1+t^2)}_{e^{-P}} \int \underbrace{\frac{1}{1+t^2}}_{e^P} \cdot \underbrace{2t}_{g} dt = \overset{\text{as above}}{(1+t^2)} \cdot \ln(1+t^2)$$

This is not the same as

$$\cancel{e^P} \int \cancel{e^P} \cdot g \cdot dt = \int 2t dt = t^2.$$

Note 3 This potentially confusing notation (which makes us want to cancel $e^{-P(t)}$ and $e^{P(t)}$) is eliminated when we solve an IVP and use correct notations.

$$\begin{aligned} \text{IVP: } & y' + p(t)y = g(t) \\ & y(t_0) = y_0 \end{aligned} \quad (8)$$

Its solution is:

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$$y(t) = e^{-P(t)} \left(\int_{t_0}^t e^{P(t_1)} g(t_1) dt_1 + y_0 \right) \quad (9a)$$

$$P(t) = \int_{t_0}^t p(t_2) dt_2 \quad (9b)$$

(Indeed, (9) has the form (7) and also satisfies the initial condition $y(t_0) = y_0$, because

$$P(t_0) = \int_{t_0}^{t_0} \dots = 0, \quad e^0 = 1,$$

$$\text{and } \int_{t_0}^{t_0} e^{P(t_1)} g(t_1) dt_1 = 0.)$$

So for solving IVPs (8) we will use solution in the form of (9).

Ex. 3 Solve IVP

$$y' - \frac{2t}{1+t^2} y = 2t$$

$$y(1) = 3$$

as in Ex. 2

Sol'n: 1) $P(t) = - \int \frac{2t_2}{1+t_2^2} dt_2 = - \ln(1+t_2^2) \Big|_1^t$

Use the given $t_0=1$.

$$= -(\ln(1+t^2) - \ln(1+1^2)) = \ln 2 - \ln(1+t^2)$$

$$2) e^{P(t)} = e^{\ln 2 - \ln(1+t^2)} = \frac{e^{\ln 2}}{e^{\ln(1+t^2)}} = \frac{2}{1+t^2}$$

$$e^{-P(t)} = \frac{1+t^2}{2}$$

$$3) \int_{t_0}^t e^{P(t_1)} g(t_1) dt_1 = \int_1^t \frac{2}{1+t_1^2} \cdot 2t_1 dt_1 =$$

Ex. 2

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$$= 2 \ln(1+t^2) \Big|_1^t = 2(\ln(1+t^2) - \ln 2).$$

4) Put together:

$$y(t) = \frac{1+t^2}{2} \cdot \left(2(\ln(1+t^2) - \ln 2) + 3 \right) \quad \swarrow y_0$$

Note: It is not always a good idea to try to simplify an answer like this because then we'll lose the easy way to see that $y(t_0) = y_0$ indeed.

Method 2 $\swarrow$ Variation of parameter
 $\leftarrow$ preferred method.

We will obtain the same formula, (6) or (7). However, the method is more general in that it can be extended to systems of several ODEs (Chap. 4).

Recall: $y_h = C_0 e^{-P(t)}$, $C_0 = \text{const}$
solves the homogeneous DE: $y_h' + p(t)y_h = 0$.

Want: Solve $y' + py = g$.

Trick: Seek $y = \underline{C(t)} e^{-P(t)}$ (10)

$\nwarrow$ Allow parameter C to vary (not be const.)

Substitute ⁽¹⁰⁾ into (5):

$$y' + py = g, \quad y = C(t)y_h$$

$$(Cy_h)' + pCy_h = g$$

$$C'y_h + \underbrace{Cy_h' + Cp y_h}_{C(y_h' + p y_h)} = g$$

$\rightarrow 0$ by the homogeneous DE

$$C'y_h = g \Rightarrow C' = \frac{g}{y_h}$$

$$C' = \frac{g}{e^{-P(t)}} = e^{P(t)} g(t), \quad \text{(choose } C_0 = 1)$$

$$\Rightarrow C = \int e^{P(t)} g(t) dt.$$

Thus, (10) becomes:

$$y(t) = e^{-P(t)} \int e^{P(t)} g(t) dt. \quad \leftarrow \text{same as (6).}$$

Again: We'll use the same method (but not method 1) in Chap. 4.

④ Important special cases of solutions of homogeneous linear DEs.

Ex. 4 Linear homogeneous DE with zero initial condition has

only the trivial solution ($y(t) \equiv 0$ for all t).

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Proof: $y' + p(t)y = 0$, $y(t_0) = 0$

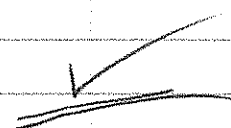
By (4),

↑
zero i.e.

$$y(t) = C \cdot e^{-P(t)}$$

@ $t = t_0$: $0 = C \cdot e^{-P(t)} \Rightarrow C = 0$

$$\Rightarrow y(t) = 0.$$



Note: In this case, $y = 0$ is the equilibrium solution.

Ex. 5 Exponential solutions

$$p(t) = -a (= \text{const}).$$

$y' = ay, \quad y(t_0) = y_0.$

 (11a)

By (9) with $g(t) = 0$:

$$P(t) = \int_{t_0}^t (-a) dt_1 = -a(t - t_0).$$

$$y(t) = e^{-P(t)} \left(\int_{t_0}^t e^{P(t_1)} \cancel{g(t_1)} dt_1 + y_0 \right)$$
$$= y_0 e^{a(t-t_0)}.$$

Thus,

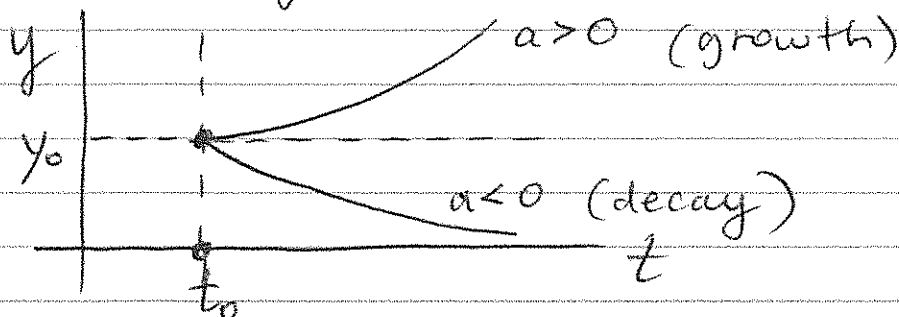
$y(t) = y_0 e^{a(t-t_0)}$

 (11b)

solves (11a).

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Note: Depending on the sign of a , the solution either exponentially grows or exponentially decays.



Ex. 6 Rational solutions: $p(t) = \frac{-a}{t}$.

$$P(t) = \int \frac{-a}{t} dt = -a \ln|t| \quad (\text{let } t > 0 \text{ for concreteness})$$

$$e^{-P(t)} = e^{a \ln t} = (e^{\ln t})^a = t^a.$$

Then

$$\boxed{y} = C \cdot e^{P(t)} = C \cdot t^a$$

is the general solution of $y' - \frac{a}{t}y = 0$.

(do not need to memorize)

Observation from Exs. 5 & 6: $y' + p(t)y = 0$

$p(t) < 0$ (const. or not) $\Rightarrow y \rightarrow \infty$ as $t \rightarrow \infty$

$p(t) > 0$ (const. or not) $\Rightarrow y \rightarrow 0$ as $t \rightarrow \infty$.

This is true for any $p(t)$, not just const or $\frac{\text{const}}{t}$.

- ⑤ Special nonhomogeneous linear DE
which is reducible to homogeneous DE.

$$y' = ay + b, \quad a, b = \text{const} (a \neq 0).$$

$$y' = a(y + \underbrace{\frac{b}{a}}_{\text{const}})$$

$$\underbrace{(y + \frac{b}{a})}' = a \underbrace{(y + \frac{b}{a})}_z$$

$$z' = az \Rightarrow z = Ce^{at} \Rightarrow$$

$$\boxed{y = -\frac{b}{a} + Ce^{at}}.$$

Q5: Suppose $a < 0$.

- What is y as $t \rightarrow \infty$?
- What is the equilibrium solution?

- ⑥ Discontinuous coefficients: Read p. 25 in book.