

Lecture 20. General properties of higher-order linear DEs

So far we've studied 2nd-order linear DEs:

$$y'' + p_1(t)y' + p_0(t)y = g(t). \quad (1)$$

We'll now state all of the facts that we've learned about (1), for a higher-order DE:

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = g(t). \quad (2)$$

We'll also present some generalizations of the earlier results.

① Existence & uniqueness

Thm. 3.5 Let in (2), $p_{n-1}(t), p_{n-2}(t), \dots, p_0(t), g(t)$ be continuous functions on $t \in (a, b)$. Also, let $t_0 \in (a, b)$. Then the IVP consisting of DE (2) and IC

$$y(t_0) = y_0, y'(t_0) = y_0', \dots, y^{(n-1)}(t_0) = y_0^{(n-1)} \quad (3)$$

has a unique solution on (a, b) .

② Properties of homogeneous linear DEs

2a Principle of linear superposition

Let $y_1(t), \dots, y_r(t)$ be solutions of the homogeneous linear DE

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = 0. \quad (4)$$

In what follows:

- We'll assume that $p_{n-1}, \dots, p_0$ are continuous on $t \in (a, b)$, so that a unique sol'n of IVP (4) + (3) exists.
- We'll drop the explicit dependence of p on t , i.e. will write p_0 for $p_0(t)$, etc.

Let $c_1, \dots, c_r$ be arbitrary constants.
Then

$$y = c_1 y_1 + \dots + c_r y_r$$

is also a sol'n of (4).

2b Fundamental sets & Wronskians

Let $y_1, \dots, y_n$ be solutions of (4).
 $\{y_1, \dots, y_n\}$ is called a FS of sol'ns if any sol'n of the IVP (4) + (3) can be written as

$$y = c_1 y_1 + \dots + c_n y_n \quad (5)$$

for some constants $c_1, \dots, c_n$.

Let us now determine a condition which determines if a set of sol'ns is a FS or not.

Suppose $\{y_1, \dots, y_n\}$ is some set of sol'ns of (4) and we are trying to construct a solution of IVP (4) + (3) in the form (5). Then:

$$y(t_0) = y_0 \Rightarrow$$

$$y_1(t_0) c_1 + y_2(t_0) c_2 + \dots + y_n(t_0) c_n = y_0$$

$$y'(t_0) = y'_0 \Rightarrow$$

$$y'_1(t_0) c_1 + y'_2(t_0) c_2 + \dots + y'_n(t_0) c_n = y'_0$$

$\vdots$

$$y^{(n-1)}(t_0) = y_0^{(n-1)} \Rightarrow$$

$$y_1^{(n-1)}(t_0) c_1 + y_2^{(n-1)}(t_0) c_2 + \dots + y_n^{(n-1)}(t_0) c_n = y_0^{(n-1)}$$

These n eqs can be written in matrix form:

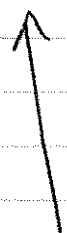
$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y'_1(t_0) & y'_2(t_0) & \dots & y'_n(t_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \\ \vdots \\ y_0^{(n-1)} \end{pmatrix} \quad (6)$$

This linear system has a unique sol'n for any values of IC, $y_0, y'_0, \dots, y_0^{(n-1)}$, (in other words, constants $c_1, \dots, c_n$ in (5) can always be found) iff the matrix in (6)

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is non-singular. That is, its determinant $\neq 0$:

$$W(t) \equiv \begin{vmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{vmatrix} \neq 0. \quad (7)$$



Wronskian.

Thus, $\{y_1, \dots, y_n\}$ is a FS iff $W(t_0) \neq 0$ at some t_0 .

As in Lecture 11 (for DE-2), one has the result:

- If the conditions of Thm. 3.5 (Existence & Uniqueness) are satisfied, then:
- either $W(t) \equiv 0$ for all $t \in (a, b)$, or
 - $W(t) \neq 0$ for all $t \in (a, b)$.

We'll now formulate a more precise result.

Thm. 3.6 (Abel's thm.)

Consider the homogeneous DE (4) and assume that conditions of Thm. 3.5 are satisfied. Then

$$W'(t) = -p_{n-1} \cdot W, \quad (8a)$$

i.e. (by Lec. 2)

$$W(t) = W(t_0) e^{-\int_{t_0}^t p_{n-1}(s) ds}, \quad (8b)$$

Note: Eq. (86) implies our earlier statement that either $W(t) \equiv 0$ on (a, b) or $W(t) \neq 0$ anywhere on (a, b) .
This is because $e^{(\text{any power})} \neq 0$.

We'll prove Abel's thm for $n=2$.
Steps of the general proof are similar.

Auxiliary facts about determinants
(stated for $n=2$, but true in general)

Fact 1 $\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0$

(If two rows (or columns) in a matrix are the same, then $\det = 0$.)

Fact 2 $\begin{vmatrix} a_1 & a_2 \\ kb_1 & kb_2 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

Fact 3 $\begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$

Fact "2+3"

$$\begin{vmatrix} a_1 & a_2 \\ kb_1+mc_1 & kb_2+mc_2 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + m \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}.$$

Fact 4 (product rule for determinants)

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}' = \begin{vmatrix} a_1' & a_2' \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_1' & b_2' \end{vmatrix}$$

(e.g., $(a_1 b_2)' = \underline{a_1' b_2} + \underline{a_1 b_2'}$)

Proof of Abel's Thm for $n=2$.

Let's rewrite DE (4) for $n=2$ as:

$$y'' = -p_1 y' - p_0 y \quad (9)$$

Differentiate the Wronskian:

$$\begin{aligned} W' &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}' \stackrel{F4}{=} \underbrace{\begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix}}_{=0 \text{ by F1}} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} \\ &\stackrel{\text{use (9)}}{=} \begin{vmatrix} y_1 & y_2 \\ -p_1 y_1' - p_0 y_1 & -p_1 y_2' - p_0 y_2 \end{vmatrix} \\ &\stackrel{F^{2+3}}{=} -p_1 \underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_W - p_0 \underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1 & y_2 \end{vmatrix}}_{=0 \text{ by F1}} \\ &= -p_1 W. \end{aligned}$$

Thus, for DE-2,

$$W' = -p_1 W. \quad (10)$$

Ex. 1 Suppose that for some FS of sol'n's of DE $y^{(4)} + y'' + y' = 0$ one has $W(1) = 2$. Find $W(10)$.

Sol'n: $y^{(4)} + 0 \cdot y''' + 1 \cdot y'' + 1 \cdot y' + 0 \cdot y = 0$
 $\Rightarrow p_3 = 0 \Rightarrow$ by Eq. (8b) $W(t) = W(1) \cdot e^{-\int_1^t p_1 ds}$
 $\quad \quad \quad \uparrow (n-1)$

$\Rightarrow W(t) = W(1)$ for all t , $\Rightarrow W(10) = W(1) = 2$.

2c FS always exist (a.k.a. How to find a FS)

Thm. 3.7 Suppose that conditions of Thm. 3.5 (Existence & Uniqueness) are satisfied for DE (4) on $t \in (a, b)$.

Then a FS of sol'n's exists for (4) on (a, b) .

Proof (for $n=3$)

Consider 3 IVP's:

$$y_1''' + p_2 y_1'' + p_1 y_1' + p_0 y_1 = 0, \quad y_1(t_0) = 1, \quad y_1'(t_0) = 0, \quad y_1''(t_0) = 0$$

$$y_2''' + p_2 y_2'' + p_1 y_2' + p_0 y_2 = 0, \quad y_2(t_0) = 0, \quad y_2'(t_0) = 1, \quad y_2''(t_0) = 0$$

$$y_3''' + p_2 y_3'' + p_1 y_3' + p_0 y_3 = 0, \quad y_3(t_0) = 0, \quad y_3'(t_0) = 0, \quad y_3''(t_0) = 1.$$

• By Thm. 3.5, their solutions y_1, y_2, y_3 exist.

• Their Wronskian

$$W(t_0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$$

$\Rightarrow$ by Abel's Thm. $W(t) \neq 0$ on (a, b) ,

$\Rightarrow \{y_1, y_2, y_3\}$ form a FS.

2d) Solutions in a FS are linearly independent

Def: Let $f_1(t), \dots, f_r(t)$ be defined on $t \in (a, b)$.
They are called linearly dependent

if there exist constants $k_1, \dots, k_r$ s.t. (11)

$$k_1 f_1(t) + k_2 f_2(t) + \dots + k_r f_r(t) = 0 \quad \text{for } \text{all } t \in (a, b),$$

and at least one of $k_1, \dots, k_r \neq 0$.

Equivalent definitions:

► Def - A Since at least one of k 's $\neq 0$,
suppose $k_1 \neq 0$. Then

$$f_1 = -\frac{k_2}{k_1} f_2 - \frac{k_3}{k_1} f_3 - \dots - \frac{k_r}{k_1} f_r$$

$$\equiv m_2 f_2 + m_3 f_3 + \dots + m_r f_r \quad (12)$$

That is, one of the functions is a linear superposition of the others for all $t \in (a, b)$.

Ex. 2 (a) $f_1 = 2, f_2 = 3 - t, f_3 = 5t + 7$
is a linearly dependent set, because:

$$2 = \frac{5}{11} (3 - t) + \frac{1}{11} (5t + 7)$$

$$\begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ f_1 & m_2 & f_2 & m_3 & f_3 \end{array}$$

or $-11 \cdot f_1 + 5 \cdot f_2 + 1 \cdot f_3 = 0$ for all t .

(b) $f_1 = 2, f_2 = 3 - t, f_3 = 5t^2 + 7$

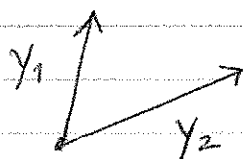
is not a lin. dep. set because one cannot

express the quadratic term in f_3 via the linear and constant terms in f_1, f_2 .

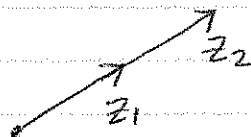
► Def-B. A set $\{f_1, \dots, f_r\}$ is called linearly independent if (11) can hold for all $t \in (a, b)$ only for $k_1 = k_2 = \dots = k_r = 0$.

Thm. 3.8 Let $\{y_1, \dots, y_n\}$ be a FS of sol'ns of DE (4). Then this set is lin. independent (w/o proof; suggested proof in HW).

Note: In Lec. 11 we saw that if $\{y_1, y_2\}$ form a FS for a DE-2, they can be compared to two non-parallel vectors in the plane:



FS $(y_1 \nparallel y_2)$



not a FS $(z_1 \parallel z_2)$

Two non-parallel vectors in the plane form a basis, meaning:

- they are linearly independent (by Def-A, this means that y_2 is not proportional to y_1 , i.e. $y_2 \nparallel y_1$).

- any vector in the plane can be written as a lin. combination of y_1, y_2 :

$$y = c_1 y_1 + c_2 y_2.$$

Now we see that for a n -th order linear homogeneous DE, solutions $y_1, \dots, y_n$ in a FS form a basis:

- $y_1, \dots, y_n$ are lin. independent;
- any solution y of the same DE is their lin. combination

$$y = c_1 y_1 + \dots + c_n y_n.$$

2e Relation between two FS's.

We'll find such a relation for $n=2$.
For $n>2$ everything will be similar.

Suppose $\{y_1, y_2\}$ and $\{\bar{y}_1, \bar{y}_2\}$ are two FSs.
Since $\{y_1, y_2\}$ is a FS, then

$$\begin{aligned}\bar{y}_1 &= a_{11} y_1 + a_{12} y_2 \\ \bar{y}_2 &= a_{21} y_1 + a_{22} y_2\end{aligned}\quad \begin{array}{l} \text{some constants} \\ (13) \end{array}$$

We can write this in a matrix form:

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad (14)$$

However, in order to make a connection to the theory to be developed in Chap. 4, we will write (13) using row-vectors $[y_1, y_2]$ and $[\bar{y}_1, \bar{y}_2]$ instead of the

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column-vectors $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$.

$$\bar{y}_1 = [y_1 \ y_2] \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}, \quad \bar{y}_2 = [y_1 \ y_2] \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$$

make this the 1st
entry (column) of a
new row-vector

$$[\text{circle} \ *] = [\text{circle} \ \text{circle}] \begin{pmatrix} \text{circle} & \text{circle} \\ \text{circle} & \text{circle} \end{pmatrix}$$

make this the 2nd column entry

$$[* \ \text{circle}] = [\text{circle} \ \text{circle}] \begin{pmatrix} \text{circle} & \text{circle} \\ \text{circle} & \text{circle} \end{pmatrix}$$

Put together:

$$[\bar{y}_1 \ \bar{y}_2] = [y_1 \ y_2] \underbrace{\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}}_A \quad (15a)$$

In general, for the n -th order linear DE (4):

$$[\bar{y}_1, \dots, \bar{y}_n] = [y_1, \dots, y_n] \cdot A \quad (15b)$$

where $\{y_1, \dots, y_n\}$ and $\{\bar{y}_1, \dots, \bar{y}_n\}$ are two FS's
and A is some constant matrix ($n \times n$).

Eqs. (15) are stated as Thm. 3.9 in textook.

Corollary 1 (stated for $n=2$; similar for $n>2$).

Let $\{y_1, y_2\}$ be a FS of a DE, and its Wronskian
is $W(t)$. Let $\{\bar{y}_1, \bar{y}_2\}$ be another FS of same DE,
and its Wronskian be $\bar{W}(t)$. Then, if

$\{y_1, y_2\}$ and $\{\bar{y}_1, \bar{y}_2\}$ are related by (15),
their Wronskians are related by:

$$\boxed{\bar{W} = W \cdot \det(A)} \quad (16)$$

Proof :

$$(15a) \Rightarrow [\bar{y}_1 \ \bar{y}_2] = [y_1 \ y_2] \cdot A$$

differentiate and use $A = \text{const matrix}$:

$$\Rightarrow [\bar{y}_1' \ \bar{y}_2'] = [y_1' \ y_2'] \cdot A$$

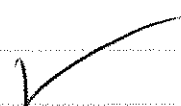
Put together as row 1 and row 2:

$$\begin{bmatrix} \bar{y}_1 & \bar{y}_2 \\ \bar{y}_1' & \bar{y}_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} A \quad (17)$$

Take the determinant of both sides
and use a formula from Linear Algebra:

$$\det(P \cdot Q) = \det P \cdot \det Q$$

to get (16),



Corollary 2 If $\{y_1, \dots, y_n\}$ and $\{\bar{y}_1, \dots, \bar{y}_n\}$
are two FS's related by (15), then
matrix A is non-singular.

Proof follows from:

- W of a FS $\neq 0$;
- $\det(A) \neq 0$ iff A is non-singular.

Ex. 3 (of formula (15) & Corollaries)

Consider DE

$$t^2 y''' + t y'' - y' = 0, \quad t > 0.$$

It is given that

$$\{y_1, y_2, y_3\} = \{1, \ln t, t^2\} \text{ is its FS.}$$

Determine if

$$\{\bar{y}_1, \bar{y}_2, \bar{y}_3\} = \{2t^2 - 1, 3, t^2 - \ln t\} \text{ is a FS.}$$

Sol'n:1) Relate $\bar{y}_1$ etc to y_1, y_2, y_3
similarly to Eq. (13):

$$\bar{y}_1 = -1 \cdot y_1 + 0 \cdot y_2 + 2 \cdot y_3$$

$$\bar{y}_2 = 3 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3$$

$$\bar{y}_3 = 0 \cdot y_1 - 1 \cdot y_2 + 1 \cdot y_3$$

Following the steps that led from (13) to (15a), we obtain in our case:

$$[\bar{y}_1 \ \bar{y}_2 \ \bar{y}_3] = [y_1 \ y_2 \ y_3] \underbrace{\begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 1 \end{pmatrix}}_A.$$

2) By Corollary 2 (see also Eq. (6)),
 $\{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ will be a FS if A is
nonsingular, i.e. if $\det A \neq 0$.

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$$\det A = \begin{vmatrix} -1 & 3 & 0 \\ 0 & 0 & -1 \\ 2 & 0 & 1 \end{vmatrix} \xrightarrow{\text{expand over 2nd row}} (-1) \cdot (-1)^{2+3} \cdot \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} \\ = 1 \cdot (-1-6) = -7 \neq 0.$$

Thus, A is nonsingular $\Rightarrow \{\bar{y}_1, \bar{y}_2, \bar{y}_3\}$ is a FS.

③ Nonhomogeneous linear DE

The n -th order DE (2) has the same superposition principle as its 2nd-order counterpart in Lecture 16:

$$\left(\begin{array}{l} \text{General sol'n} \\ \text{of a nonhomog.} \\ \text{linear DE} \end{array} \right) = \left(\begin{array}{l} \text{particular} \\ \text{sol'n} \end{array} \right) + \left(\begin{array}{l} \text{general sol'n} \\ \text{of homogen.} \\ \text{DE} \end{array} \right)$$

$$y = y_p + y_c.$$