Lecture 22 Systems of 1st-order linear DEs : Introduction

① Motivation

We've mostly studied 1st-order & 2nd-order DEs for one variable y. They describe the rate of change or acceleration of one quantity:

$$\frac{dy}{dt} = \dots \quad \leftarrow \text{DE-1, rate of change}$$

$$\frac{d^2y}{dt^2} = \dots \quad \leftarrow \text{DE-2, acceleration}$$

However, in real world, things rarely change in isolation from one another. E.g., quantity y may depend on z and vice versa. Then we will have:

$$\frac{dy}{dt} = f_1(y, z)$$

$$\frac{dz}{dt} = f_2(y, z)$$

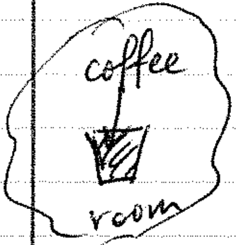
Ex. 1 Set up a system of 1st-order DEs describing heating/cooling of a house with two "rooms".

Preliminaries

In Lec. 1 (Ex. 2(b)) and HW 4 we

considered Newton's Cooling Law:

$$\frac{dT}{dt} = -K(T - T_{room}) = K(T_{room} - T)$$



This law is actually a corollary of the balance of heat (which is similar to balance equation for mixing problems in Lec. 4):

"amount of heat" (i.e., energy)

$$\frac{dQ}{dt} = \underbrace{\begin{pmatrix}\text{Heat}\\ \text{flow}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{Heat}\\ \text{flow}\\ \text{out}\end{pmatrix}}_{m\cdot(T_{external} - T)} \qquad (1)$$

some constant

$\underline{T_{ext} > T} \Rightarrow$ flow is <u>in</u> (>0)

$\underline{T_{ext} < T} \Rightarrow$ flow is <u>out</u> (<0)

To supplement this equation, we need to relate "amount of heat" with temperature. From thermodynamics:

$$\underbrace{\Delta T} = c \cdot \underbrace{\Delta Q}$$

change in temperature caused by ΔQ

heat capacity (units = 1°F/BTU)

change in amount of heat (by adding or subtracting heat)

usually measured in BTU (British thermal unit = amount of heat needed to increase temp. of 1 lb of H_2O by 1°F).

Then (1) becomes

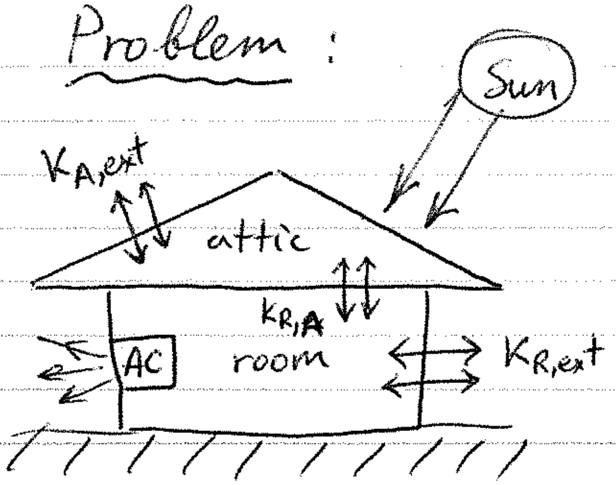
$$\frac{1}{c}\cdot\frac{dT}{dt} = m\,(T_{ext} - T)$$

$$\Rightarrow \frac{dT}{dt} = \underbrace{(cm)}_{\uparrow \atop \text{"K" of Newton's Cooling Law}} \cdot (T_{ext} - T) \qquad (2a)$$

or

$$\boxed{\frac{dQ}{dt} = \left(\frac{K}{c}\right)\cdot\left(T_{ext} - T\right)} \qquad (2b)$$

Problem :



- Sun heats up the attic in a house at the rate of Fsun BTU/hr.
- The outside temperature is $T_{ext} = 100°F$.
- The constants K in Newton's Law are:

$K_{A,ext}$ — between Attic & Outside
$K_{R,ext}$ — between Room & Outside
$K_{R,A}$ — between Room & Attic

- The air conditioner in the room removes

F_{AC} BTU/hr of heat.

- The heat capacity of the air (in room & attic) is c.

Set up a system of equations for the evolution of the temperatures T_A & T_R in the Attic and Room.

Sol'n :

Balance eq. for Attic:

$$\frac{dQ_A}{dt} = \underset{\substack{\uparrow \\ \text{heat flow} \\ \text{from Sun}}}{F_{sun}} + \underbrace{\frac{K_{A,ext}}{c}\left(T_{ext}-T_A\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Attic \& Outside}}} + \underbrace{\frac{K_{R,A}}{c}\left(T_R-T_A\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Attic \& Room}}}$$

Using $dQ_A/dt = d(T_A/c)/dt$ we get:

$$\boxed{\frac{dT_A}{dt} = c \cdot F_{sun} + K_{A,ext}\left(T_{ext}-T_A\right) + K_{R,A}\left(T_R-T_A\right)} \quad (3a)$$

Balance for Room:

$$\frac{dQ_R}{dt} = \underset{\substack{\nearrow \\ \text{heat removal} \\ \text{by AC}}}{-F_{AC}} + \underbrace{\frac{K_{R,ext}}{c}\left(T_{ext}-T_R\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Room \& Outside}}} + \underbrace{\frac{K_{R,A}}{c}\left(T_A-T_R\right)}_{\substack{\text{heat exchange} \\ \text{between} \\ \text{Room \& Attic}}}$$

Using $dQ_R/dt = d(T_R/c)/dt$ we get:

$$\boxed{\frac{dT_R}{dt} = -c \cdot F_{Ac} + K_{R,ext}(T_{ext} - T_R) + K_{R,A}(T_A - T_R)} \quad (3b)$$

Eqs. (3a) & (3b) form the desired system of eqs.

Example 1 in book (Sec. 4.1) considers a mixing problem in two connected tanks, leading to a similar system.

In the most general case, a system of two linear, 1st-order DEs has the form:

$$\begin{aligned} y_1' &= p_{11}(t)\,y_1 + p_{12}(t)\,y_2 + g_1(t) \\ y_2' &= p_{21}(t)\,y_1 + p_{22}(t)\,y_2 + g_2(t), \end{aligned} \quad (4)$$

or in matrix form:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad (5a)$$

or, in vector-matrix notations:

$$\vec{y}\,' = \underset{\substack{\uparrow \\ \text{matrix}}}{P(t)}\,\vec{y} + \vec{g}(t). \quad (5b)$$

For n lin. DEs, the system has the same form (5b).

② Auxiliary facts about algebra and Calculus of vectors & matrices.

$\boxed{2a}$ Algebra

! → 0. (A is invertible (i.e. A^{-1} exists)) $\Leftrightarrow$ ($\det A \neq 0$).

1. $\det(AB) = \det(A) \cdot \det(B)$

2. $\det(\underset{\substack{\uparrow \\ \text{scalar}}}{a} \cdot \underset{\substack{\uparrow \\ \text{matrix} \\ n \times n}}{A}) = a^n \cdot \det(A)$

3. Addition, scalar multiplication etc. of matrices — intuitive (see p. 216)

not of det's!!!

4. Let $\vec{c}_1, \vec{c}_2, \dots \vec{c}_r$ be $n \times 1$-columns and B be a $n \times n$ matrix.
Let
$$\underset{n \times r}{C} = [\vec{c}_1, \vec{c}_2, \dots \vec{c}_r].$$
Then
$$BC = [B\vec{c}_1, B\vec{c}_2, \dots B\vec{c}_r]$$
(i.e., matrix B multiplies each column of matrix C).

5. VERY IMPORTANT PROPERTY:

Let $A = [\vec{A}_1, \vec{A}_2, \dots \vec{A}_n]$.

↑ $n \times n$ matrix — $n \times 1$ vectors.

Let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$\boxed{\begin{aligned} A\vec{x} &\equiv [\vec{A_1}, \vec{A_2}, \vec{A_3}, \dots \vec{A_n}]\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} \\ &= x_1\vec{A_1}+x_2\vec{A_2}+\dots+x_n\vec{A_n}\end{aligned}} \quad (6)$$

Ex. 2 Verify (6) for $n=2$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}\cdot x_1 + a_{22}x_2 \end{pmatrix} = x_1\underbrace{\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}_{\vec{A_1}} + x_2\underbrace{\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}_{\vec{A_2}}$$

$$\underbrace{\phantom{a_{21}}}_{\vec{A_1}} \quad \underbrace{\phantom{a_{22}}}_{\vec{A_2}}$$

✓

$\boxed{26}$ Calculus.

1. Derivatives – just differentiate each entry:

$$A' \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}' = \begin{pmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{pmatrix}.$$

Note: This should not be confused with the derivative of a determinant (see p. 20-5, Fact 4).

2. Derivatives of sum and product – just as for scalars (p. 219).

3. Integration – just integrate each term.

Ex. 3 Find $\int A(t)dt$, $A(t) = \begin{pmatrix} e^{3t} & 2t \\ 0 & -1 \end{pmatrix}$.

Sol'n: $\int A(t)dt = \begin{pmatrix} \int e^{3t}dt & \int 2t\,dt \\ \int 0\,dt & \int (-1)dt \end{pmatrix}$

$$= \left(\begin{array}{c|c} \frac{1}{3}e^{3t} + c_{11} & t^2 + c_{12} \\ 0 + c_{21} & -t + c_{22} \end{array}\right) = \begin{pmatrix} \frac{1}{3}e^{3t} & t^2 \\ 0 & -t \end{pmatrix} + \underbrace{\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}}$$

the integration "constant" becomes a constant matrix