Lecture 25. Nonhomogeneous linear systems of DEs with constant coefficients: Method of variation of parameters.

We will show how to solve a nonhomog. linear system of DEs

$$\vec{y}\,' = P(t)\,\vec{y} + \vec{g}(t), \qquad (1)$$

where $\vec{y}, \vec{g}$ are $n\times 1$ vectors and $P(t)$ is an $n\times n$ matrix,
<u>provided</u> that we know the complementary sol'n y_c, i.e.

$$\vec{y}_c\,' = P(t)\,\vec{y}_c\,. \qquad (2)$$

Since one can solve (2) by the method of Lecture 24 for $P(t) =$ const matrix, our examples will consider ~~that~~ that case only. However, the same method works for $P(t) \neq$ const.

Suppose we KNOW a FS of solutions of (2): $\{\vec{y}_1(t), \dots \vec{y}_n(t)\}$. Then:

$$\vec{y}_1\,' = P\,\vec{y}_1, \quad \vec{y}_2\,' = P\vec{y}_2, \quad \dots \quad \vec{y}_n\,' = P\vec{y}_n \qquad (3a)$$

Combining these into columns, we have:

$$[\vec{y}_1\,', \vec{y}_2\,', \dots \vec{y}_n\,'] = [P\vec{y}_1, P\vec{y}_2, \dots P\vec{y}_n], \text{ or}$$

$$\underbrace{[\vec{y}_1, \dots \vec{y}_n]}_{\Psi(t)}\,' = P\,\underbrace{[\vec{y}_1, \dots \vec{y}_n]}_{\Psi(t)} \qquad (3b)$$

fundamental matrix (Lec. 23)

or $$\boxed{\underset{n\times n}{\Psi'} = \underset{n\times n}{P}\,\underset{n\times n}{\Psi}} \qquad (3c)$$

Let us recall that the general solution of (2) can be represented as

$$\vec{y_c} = c_1\vec{y_1} + c_2\vec{y_2} + \ldots + c_n\vec{y_n}$$

VERY IMPORTANT FACT, p. 22-6

$$= [\vec{y_1}, \vec{y_2}, \ldots \vec{y_n}]\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \Psi\vec{c}, \qquad (4)$$

where $\vec{c}$ = const vector.

Then, similarly to the Method of Variation of parameters for a scalar DE, we look for a solution of (1) in the form

$$\vec{y} = \Psi(t)\,\vec{u}(t), \qquad (5)$$

where $\Psi(t)$ is a fundamental matrix of (2) (which solves (3c)) and $\vec{u}(t)$ is a vector to be found.

Substitute (5) into (1):

$$(\Psi\vec{u})' = P(\Psi\vec{u}) + \vec{g} \quad \Rightarrow$$

$$\cancel{\Psi'\vec{u}} + \Psi\vec{u}' = \cancel{(P\Psi)\vec{u}} + \vec{g} \quad \Rightarrow$$

same by (3c)

$$\Psi\vec{u}' = \vec{g} \quad \Rightarrow \quad \vec{u}' = \Psi^{-1}\vec{g}$$

$$\Rightarrow \boxed{\vec{u}(t) = \vec{u}(t_0) + \int_{t_0}^{t} \Psi^{-1}(s)\,\vec{g}(s)\,ds} \qquad (6)$$

Therefore, a particular sol'n of (1) is:

$$\vec{y}(t) = \Psi(t)\left[\vec{u}(t_0) + \int_{t_0}^{t} \Psi^{-1}(s)\,\vec{g}(s)\,ds\right] \quad (7).$$

Note 1: In (7), $\vec{u}(t_0)$ can be found to satisfy a given initial condition as follows:

$$\vec{y}(t_0) = \Psi(t_0)\left[\vec{u}(t_0) + \cancel{\int_{t_0}^{t_0} \cdots}^{\,0}\right] \Rightarrow$$

$$\vec{y}(t_0) = \Psi(t_0)\,\vec{u}(t_0) \Rightarrow$$

$$\boxed{\vec{u}(t_0) = \Psi^{-1}(t_0)\cdot\vec{y}(t_0)}. \quad (8)$$

Note 2: If no initial condition is specified, then in (7) one can take the indefinite integral.

Note 3: In (6), (7), and (8) we know that $\Psi(t_0)^{-1}$ exists because $\Psi(t_0)$ is a fundamental matrix and hence $W(t_0) = \det\Psi(t_0) \neq 0$.

Ex. 1 Use the method of variation of parameters to solve the given IVP.

$$\vec{y}\,' = \begin{pmatrix} -2 & 5 \\ 1 & 2 \end{pmatrix}\vec{y} + \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \vec{y}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Sol'n: 1) Find a fundamental matrix.
a) Find eigenvalues:

skip details

$$\begin{vmatrix} -2-\lambda & 5 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm 3.$$

b) Find eigenvectors:

$\underline{\lambda = 3}$ $\quad \begin{pmatrix} -2-3 & 5 \\ 1 & 2-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 - x_2 = 0$

$\Rightarrow \quad \vec{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

$\underline{\lambda = -3}$ $\quad \begin{pmatrix} -2+3 & 5 \\ 1 & 2+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + 5x_2 = 0$

$\Rightarrow \quad \vec{x_2} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

c) $\quad \Psi(t) = [e^{3t}\vec{x_1},\ e^{-3t}\vec{x_2}] = \begin{pmatrix} e^{3t} & -5e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix}.$

2) Find $\vec{u}(t_0)$ from (8).
(This step can be skipped if you are not given an initial condition.)

$$\vec{y}(\cancel{t_0}^{\,0}) = \Psi(\cancel{t_0}^{\,0}) \cdot \vec{u}(t_0)$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix} \cancel{\vec{u}_0}^{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

We can use Ψ^{-1} from p. 217 of textbook or find c_1, c_2 by inspection:

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -5 \\ 1 \end{pmatrix} \Rightarrow c_1 = \tfrac{1}{2},\ c_2 = -\tfrac{1}{2}.$$

So $\quad \vec{u_0} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}.$

3) Find the integral term in (7).

a) Find $\psi^{-1}(s) = \begin{pmatrix} e^{3s} & -5e^{-3s} \\ e^{3s} & e^{-3s} \end{pmatrix}^{-1}$

formula for A^{-1} from p.217

$$\overset{*}{=} \frac{1}{e^{3s}\cdot e^{-3s} - (-5e^{-3s})\cdot e^{3s}} \cdot \begin{pmatrix} e^{-3s} & 5e^{-3s} \\ -e^{3s} & e^{3s} \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} e^{-3s} & 5e^{-3s} \\ -e^{3s} & e^{3s} \end{pmatrix}$$

b) Find $\psi^{-1}(s)\vec{g}(s) = \frac{1}{6}\begin{pmatrix} e^{-3s} & 5e^{-3s} \\ -e^{3s} & e^{3s} \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} 3/2\, e^{-3s} \\ -1/2\, e^{3s} \end{pmatrix}.$$

c) Integrate:

integrate each entry

$$\int_{t_0 \to 0}^{t} \begin{pmatrix} 3/2\, e^{-3s} \\ -1/2\, e^{3s} \end{pmatrix} ds \overset{*}{=} \begin{pmatrix} -\frac{1}{2} e^{-3s} \Big|_0^t \\ -\frac{1}{6} e^{3s} \Big|_0^t \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 3(e^{-3t}-1) \\ e^{3t}-1 \end{pmatrix}.$$

4) Put everything together (as per Eq. (7)):

$$\vec{y}(t) = \left(\begin{array}{c|c} e^{3t} & -5e^{-3t} \\ e^{3t} & e^{-3t} \end{array}\right)\left[\begin{pmatrix} \frac{1}{2} \\ -1/2 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 3e^{-3t}-3 \\ e^{3t}-1 \end{pmatrix}\right]$$

$$= \left(\begin{array}{c|c} e^{3t} & -5e^{-3t} \\ e^{3t} & e^{-3t} \end{array}\right)\begin{pmatrix} 1 - \frac{1}{2}e^{-3t} \\ -\frac{1}{3} - \frac{1}{6}e^{3t} \end{pmatrix} = \ldots$$

(can multiply out if needed...)