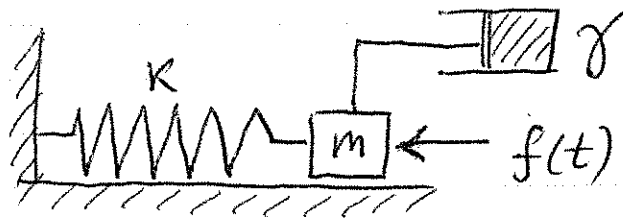


Lecture 26 Laplace transform (LT): Motivation and Introduction.

Although LT can be developed to systems of linear differential eqs, we'll consider it only for the scalar, second-order DEs.

A necessary condition to be able to apply LT is that the DE had constant coefficients.

① Motivation



- Suppose we have a vibrating system whose model we know to be

$$m y'' + \gamma y' + k y = f(t), \quad (1)$$

but we do not know m, γ, k (say, the system is in a black box).

- Suppose we drive the system with a known force $f(t)$ and measure its response $y(t)$.
- From this response, the form $f(t)$, and the known form of (1) (oscillator with damping), can we compute the parameters m, γ, k of the system?

Answer 1: It is possible, but very tedious and technically complex to do if we use

solution techniques developed so far
(in Lec. 15 & 19).

Answer 2: It is very easy if we use the
method of Laplace transform.

→ So the main usefulness of LT is that it allows
one to deduce the system properties by
measuring its response to a known input (force).

(2) "Bird's view" idea of LT

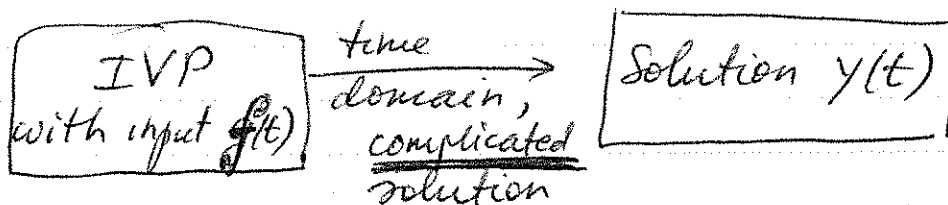
Instead of working with functions in ^{the} time
domain (i.e., $y(t)$ or $f(t)$ or $g(t)$),
LT works in a transform domain
described by some variable s .

So, Laplace transform does this:

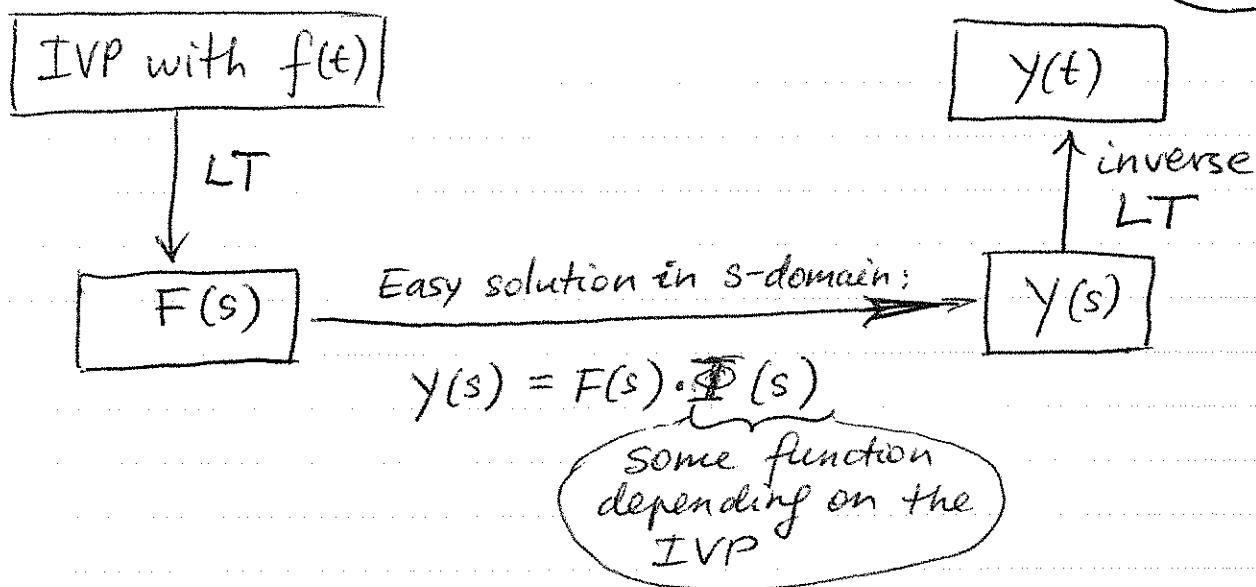
$$f(t) \xrightarrow{LT} F(s)$$

(we will illustrate details shortly).

Thus, instead of solving an IVP in the
time domain:



we go in a round-about way:



Thus, we need to learn 3 steps!

- 1) How to find LT of $f(t)$;
(in particular, how to find $\Phi(s)$)
- 2) How to solve the IVP in the s -domain;
- 3) How to transform the "solution" $y(s)$ from the s -domain to the t -domain.
(back)

③ Definition of LT

Def. Let $f(t)$ be a function defined on $t \in [0, \infty)$, Then its LT:

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^{\infty} f(t) e^{-st} dt, \quad (2)$$

provided that this improper integral exists (Calc. 2)

Note 1: Unlike in Lec. 25, where " s " was just another notation for the time variable, here " s " is a completely different variable not related to time at all!

Note 2 Recall from Calc. 2:

$$\int_0^{\infty} f(t) e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T f(t) e^{-st} dt \quad (3)$$

(If the integral exists, it is said to converge, otherwise it is said to diverge.)

Ex. 1 Find the LT, if it exists, of:

(a) $f(t) e^{at}$; (b) $f(t) = t$; (c) $f(t) = e^{t^2}$.

Sol'n: (a) $F(s) = \int_0^{\infty} e^{at} \cdot e^{-st} dt = \int_0^{\infty} e^{(a-s)t} dt$
 $= \lim_{T \rightarrow \infty} \int_0^T e^{(a-s)t} dt = \lim_{T \rightarrow \infty} \frac{e^{(a-s)T} - 1}{a-s} = \begin{cases} \infty, & a > s \\ \frac{1}{s-a}, & a < s. \end{cases}$

Thus, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a.$

(b) $F(s) = \int_0^{\infty} t e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T t e^{-st} dt$

$= \lim_{T \rightarrow \infty} \left[\begin{array}{l} T^2/2, \quad s=0 \\ \frac{1}{s^2} \left(-e^{-sT}(1+sT) + 1 \right), \quad s>0 \end{array} \right]$

this $\rightarrow 0$ as $T \rightarrow \infty$
 because e^{-sT} "wins" over T .

$= \frac{1}{s^2} (0+1) = \frac{1}{s^2}, \quad s > 0.$

Thus, $\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s > 0.$

int. by
parts
(Calc. 2)

$$\begin{aligned}
 (c) \quad F(s) &= \int_0^{\infty} e^{t^2} \cdot e^{-st} dt \\
 &= \lim_{T \rightarrow \infty} \int_0^T e^{t^2 - st} dt.
 \end{aligned}$$

This integral cannot be computed as an "elementary" function (i.e. that studied in Calculus). However, as $t \rightarrow T \rightarrow \infty$,

$t^2 > st$ for any given value of s .

Therefore, $e^{t^2 - st} \approx e^{(\rightarrow +\infty)} \rightarrow \infty$,

and so the integral in this case diverges (does not exist).

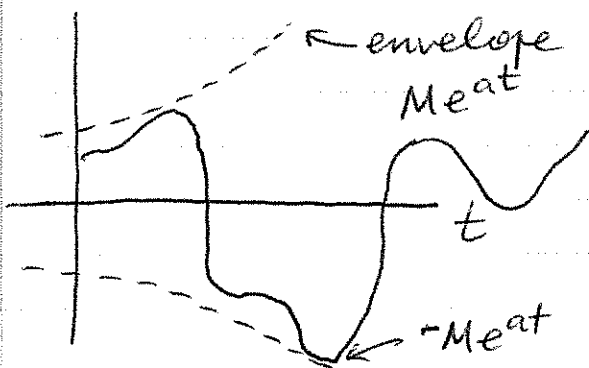
Thus, $\mathcal{L}\{e^{t^2}\}$ does not exist. //

④ Existence of LT

Ex. 1(c) motivates the following definition:

Def: Function $f(t)$ defined on $t \in [0, \infty)$ is called exponentially bounded if there are constants $M \geq 0$ and a s.t.

$$|f(t)| \leq M e^{at} \text{ for } \underline{\underline{all}} \ t > 0.$$



Basically, this means that for $t \rightarrow \infty$, we do not want $f(t)$ to grow faster than e^{at} for some a .

In Ex. 1(c), $f(t) = e^{t^2}$ is not exponentially bounded, as it grows faster than e^{at} for any given a .

Thm. 5.1 (Existence of LT).

Let $f(t)$ be piecewise continuous and exponentially bounded on $t \in [0, \infty)$. Then its LT

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

exists for all $s > a$.

(Examples: Ex. 1(a); 1(b) ($a=0$).)

In what follows we will always assume that $f(t)$ is such that its LT exists.

⑤ Linear superposition principle for LT

Thm. 5.2 Let $f_1(t)$ & $f_2(t)$ be such that their LT exist for $s > a$ (for some a). Let c_1, c_2 be arbitrary constants. Then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

⑥ The Inverse Laplace Transform and the issue of Uniqueness

Thm. 5.3 Suppose that $f_1(t), f_2(t)$ are continuous on $[0, \infty)$. Let $F_1(s), F_2(s)$ be their respective LT, which we assume exist for $s > a$. Then

$$\left(\begin{array}{l} f_1(t) = f_2(t) \\ \text{for all } t > 0 \end{array} \right) \Leftrightarrow \left(\begin{array}{l} F_1(s) = F_2(s) \\ \text{for all } s > a \end{array} \right).$$

That is, if LT's of two continuous functions are equal, so are the original functions.

Note: If f_1, f_2 are only piecewise continuous, then they are equal at all points of their continuity (see book, p. 325).

This Thm. motivates a method of finding the inverse of LT: build a look-up table of LT's:

function	LT
$f_1(t)$	$F_1(s)$
$f_2(t)$	$F_2(s)$
$\vdots$	$\vdots$

Since by Thm. 5.3 there is a unique correspondence between $f(t)$ and its $F(s)$, then from the table we can find $f(t)$ given a $F(s)$.

So far we have, from Ex. 1(a), (b)

function	LT
e^{at}	$\frac{1}{s-a}, \quad s > a$
t	$\frac{1}{s^2}, \quad s > 0$

Let us show how it can be used.

Ex. 2 Let $F(s) = \frac{2s}{s^2-1}$, $s > 1$.

What is $f(t)$? (I.e., find the inverse LT of $\frac{2s}{s^2-1}$.)

Sol'n: The given $F(s)$ is not in our table. However, we can use Partial Fraction Expansion (review it from Calc. 2!) to write:

$$\frac{2s}{s^2-1} = \frac{1}{s-1} + \frac{1}{s+1}$$

Again, Review the method of Part. Fractions on how to find these coefficients.

Now, both $\frac{1}{s-1}$ & $\frac{1}{s+1}$ are in our table:

it is $\frac{1}{s-a}$ with $a=1$ & $a=-1$.

Then, denoting the inverse LT by $\mathcal{L}^{-1}$ and using the linear superposition principle (Thm. 5.2), we have:

$$\mathcal{L}^{-1} \left\{ \frac{1}{s-1} + \frac{1}{s+1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

from table $\rightarrow e^t + e^{-t}$.

$$\text{Thus, } \mathcal{L}^{-1} \left\{ \frac{2s}{s^2-1} \right\} = e^t + e^{-t} //$$