

Lecture 3 General properties of solutions of linear DEs

Notations:

don't need to state at beginning of lecture

old

$$\begin{cases} y' + py = 0 & (H) \leftarrow \text{homogeneous DE} \\ y' + py = g & (NH) \leftarrow \text{non-homogeneous DE} \end{cases}$$

NEW

" $t \in (a, b)$ " means " $a < t < b$ "

"belongs to"

"is in"

$(\quad)$
a t b

① Existence and uniqueness of solution

Thm. 2.1 (p. 17)

Let $p(t), g(t)$ be continuous for all $t \in (a, b)$.
Let also $t_0 \in (a, b)$. Then the IVP

$(\quad)$
a t_0 b

$$\begin{aligned} y' + p(t)y &= g(t) \\ y(t_0) &= y_0 \end{aligned} \quad (1)$$

has a unique sol'n for all $t \in (a, b)$.

Proof: 1) Existence ("has")

This follows from formula (9) of Lec. 2, which presented an explicit solution of IVP (1):

$$y(t) = e^{-P(t)} \left(\int_{t_0}^t e^{P(t_1)} g(t_1) dt_1 + y_0 \right) \quad (2a)$$

$$P(t) = \int_{t_0}^t p(t_2) dt_2. \quad (2b)$$

Since $p(t)$ is continuous on (a, b) , then so are $P(t)$ and $e^{\pm P(t)}$.

Similarly, since $g(t)$ is continuous, then so is $\int_{t_0}^t e^{P(t_1)} g(t_1) dt_1$, i.e. it exists. ✓

2) Uniqueness

The method of proof used below is commonly used in many other theorems.

Proof (by contradiction):

Suppose $y_1(t)$ & $y_2(t)$ are two solns of IVP (1):

$$\begin{cases} y_1' + p y_1 = g \\ y_2' + p y_2 = g \end{cases}, \quad \begin{cases} y_1(t_0) = y_0 \\ y_2(t_0) = y_0. \end{cases}$$

Subtract the 2nd eqs from the 1st and denote

$$w = y_1 - y_2 :$$

$$(y_1 - y_2)' + p(y_1 - y_2) = \cancel{g} - \cancel{g}^0, \quad y_1(t_0) - y_2(t_0) = \cancel{y_0} - \cancel{y_0}^0$$

$$\begin{cases} w' + p w = 0 \\ w(t_0) = 0 \end{cases}$$

← Homogeneous DE with zero IC.

3-3

In Ex. 4 of Lec. 2 we showed that the only solution of this IVP is

$$w(t) = 0 \text{ for all } t, \Rightarrow$$

$$y_1(t) - y_2(t) = 0 \Rightarrow y_1(t) = y_2(t) \text{ for all } t.$$

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Q: What if $p(t)$ and/or $g(t)$ are not continuous on (a, b) ?
Does it mean that $y(t)$ does not exist?

A: It depends; see the next Example.

Ex. 1 (a) The IVP $y' = \frac{1}{t} y, y(-2) = 3$
 $\uparrow$
 $-p(t)$ not continuous on $(-2, 1)$

has the solution $y = -\frac{3}{2}t$ (similar to Ex. 6 of Lec. 2; also can verify by substitution).

Thus, here $y(t)$ exists and is continuous.

(b) The IVP $y' = \ominus \frac{1}{t} y, y(-2) = 3$
 $\uparrow$
 (difference from (a))

has the solution $y = -\frac{6}{t}$, which is not cont.

Moral: No general conclusion about the existence of solution can be made if $p(t)$ & $g(t)$ are not continuous.

② Superposition principle (not IVP)

The goal of solving a DE is to find as many of its solutions as possible.
(This becomes especially relevant and challenging for higher-order DEs, which contain y'' , y''' , etc.)

Hence, Q: If we know some two solutions of a DE, can we use them to construct a third solution?

A: Affirmative for linear DEs.

Thm. (Superposition principle).

(a) Let $y_1(t)$, $y_2(t)$ be two solutions of the homogeneous DE

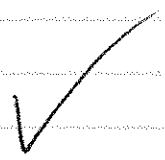
$$y' + py = 0 \quad (H)$$

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Then $y_3 = c_1 y_1 + c_2 y_2$, where $c_{1,2} = \text{const}$, is another solution of (H).

Proof (similar to Proof of Uniqueness of Thm. 2.1):

$$\begin{aligned} & \left. \begin{aligned} c_1 \cdot (y_1' + p y_1) &= 0 \\ c_2 \cdot (y_2' + p y_2) &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} c_1 y_1' + c_1 p y_1 &= 0 \\ c_2 y_2' + c_2 p y_2 &= 0 \end{aligned} \\ \Rightarrow & \begin{aligned} & (c_1 y_1)' + p(c_1 y_1) = 0 \\ & (c_2 y_2)' + p(c_2 y_2) = 0 \end{aligned} \\ \Rightarrow & \underbrace{(c_1 y_1 + c_2 y_2)'}_{y_3} + p \underbrace{(c_1 y_1 + c_2 y_2)}_{y_3} = 0 \end{aligned}$$



(b) Let y_1 satisfy (H) and y_2 satisfy

$$y_2' + py_2 = g \quad (\text{NH})$$

Then $y_3 = cy_1 + y_2$, where $c = \text{any const}$, is also a solution of (NH). ↑
non-homogeneous

Proof: At Home, similar to part (a). ✓

Corollary

Suppose one knows one sol'n, $y_p(t)$ of (NH). To find the general sol'n of (NH), do:

↑
(particular sol'n)

1) Solve (H): $y_h' + py_h = 0$
and find any y_h .

2) General sol'n of NH:

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$$y = cy_h + y_p, \\ c = \text{const}$$

(3)

Ex. 2

By inspection, it is easy to see that a sol'n of

$$y' = y - t + 1$$

is $y_p = t$. Find the general sol'n.

Sol'n:

Find y_h s.t. $y_h' = y_h$.
By Lec. 2, $y_h = e^t$.

Then, by Corollary, the general sol'n is

$$y = ce^t + t.$$

Note 1 If y_1, y_2 are two solutions of (NH), then $(y_1 + y_2)$ is not a solution of (NH).
Proof - at home.

Note 2 Superposition Principle (neither (a) nor (b)) for nonlinear DEs is not valid.

(This is why linear DEs are so important. In contrast, for nonlinear DEs, knowing two particular solutions gives no further information about the general sol'n.)

Ex. 3 Consider $y' = y^2 - t^2 + 1$. (4)
 By inspection, $y = t$ is a particular sol'n.
 Also, $y_h = -1/t$ solves $y_h' = y_h^2$
 (will show this later).
 Show that $y = t - c/t$ is not a sol'n of (4) unless $c = 0$.

Sol'n: Lhs: $y' = 1 + \frac{c}{t^2}$

Rhs: $(t - \frac{c}{t})^2 - t^2 + 1 = \cancel{t^2} - 2t \cdot \frac{c}{t} + \frac{c^2}{t^2} - \cancel{t^2} + 1$

So: $\cancel{1} + \frac{c}{\cancel{t^2}} \stackrel{?}{=} -2c + \frac{c^2}{\cancel{t^2}} + \cancel{1}$
 $\uparrow$ not equal for $c \neq 0$.