

Lecture 4 Some applications of linear DEs

① Population models (Sec. 2.4)

Simplest model:

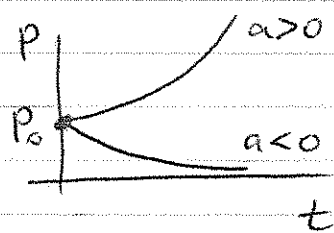
$$\boxed{\text{Rate of population change}} = \boxed{\text{Rate of population increase}} - \boxed{\text{Rate of population decrease}}$$

$$\frac{dP}{dt} = r_b \cdot P - r_d \cdot P$$

$\uparrow$ births (assuming unlimited supply of resources & no competition) $\uparrow$ mortality

$$\frac{dP}{dt} = \underbrace{(r_b - r_d)}_{a = \text{const.}} P \quad (1)$$

Solution is: $P(t) = P(0) e^{at}$
(Ex. 5 in Lec. 2).



See also Ex. 1, 2 in Sec. 2.4.

② Radioactive decay

Let $Q(t)$ be amount of radioactive material at time t .

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It only decays ($r_i = 0$); let $r_d \equiv k$. Then

$$\frac{dQ}{dt} = -kQ$$

$$\Rightarrow Q(t) = Q(0) e^{-kt} \quad (2)$$

Ex. 1 Suppose that 40% of some radioactive material decays in 4 years. Find the time that it takes 50% of the initial material to decay.

Sol'n: 1) $Q(t) = Q_0 \cdot e^{-kt}$
 $\uparrow$
 $Q(0)$

We know $Q(4) = 0.6Q_0$, $\Rightarrow$

$$0.6Q_0 = Q_0 \cdot e^{-k \cdot 4}$$

$$\ln 0.6 = -4k, \quad k = -\frac{1}{4} \ln 0.6 \approx 0.128$$

2) If $Q(T) = 0.5Q_0$, $\Rightarrow$

$$0.5Q_0 = Q_0 e^{-kT}$$

$$\ln \frac{1}{2} = -kT \Rightarrow \boxed{T = \frac{\ln 2}{k}} \quad (3)$$

$(-\ln 2)$

half-life of material

For $k \approx 0.128$, $T \approx 5.4$ years.

Radiocarbon dating: This is a technique to measure the age of an archeological object based on the amount of Carbon-14 in it.



Some details of the technique:

- ${}^{14}\text{C}$ is a radioactive isotope of carbon produced when the stable (non-decaying) isotope ${}^{12}\text{C}$ is exposed to cosmic radiation:



- Once produced, ${}^{14}\text{C}$ has half-life of 5730 years.
- As the first approximation, one can assume that the ratio of ${}^{14}\text{C}$ to ${}^{12}\text{C}$ atoms in the atmosphere is known. It has the order of $1:10^{12}$ (one per trillion), although its value significantly varies depending on the location and the time moment back in the past. (See more on this in the Wikipedia article.)

A look-up table of values of this ratio has been created over the years, based on some math models and on comparison of the radiocarbon dating results with results produced by other dating methods. A lot of work went into creation of this table.

- A living organism absorbs all carbon from the atmosphere and hence continues to maintain the same ratio of ^{14}C to ^{12}C as in the atmosphere throughout its lifetime. When the organism dies, the exchange of carbon with the atmosphere stops, and the amount of ^{14}C begins to decrease according to Eq. (2) (p. 4-2). Hence the ratio

$$\frac{Q_{14}(t)}{Q_{12}(t)} = \frac{Q_{14}(0)e^{-kt}}{Q_{12}(0)}$$

$\uparrow$ stays const; ^{12}C not radioactive

exponentially decays.

- A modern technique called accelerator mass spectrometry is capable of counting numbers of ^{12}C and ^{14}C and hence to determine the ratio $Q_{14}(t)/Q_{12}(t)$. (In 1950-1970s, researchers used different counting techniques.)

Thus, knowing:

$$\frac{Q_{14}(t)}{Q_{12}(t)}, \quad \frac{Q_{14}(0)}{Q_{12}(0)} \text{ (from look-up table)}, \text{ and } k,$$

one can find t — the age of the object.

This is further explored in some HCV problems.

③ Mixing problems (Sec. 2.3)

Ex. 2 The water in a shallow polluted lake initially contains 1 lb of mercury salts per 100,000 gallons of water.

Initially, there are 500,000 gallons of water in the lake.

Polluted water is pumped out of the lake at a rate 1000 gal/h and fresh water is pumped in at a rate 1500 gal/h.

What is the concentration of mercury salts in the lake when its volume reaches 600,000 gal?

Notations:

m = amount of mercury in lake;

W = amount of water

K = concentration of mercury.

Relation between K, m, W :

→ (Obviously, $m = K \cdot W$.)

Given: $W(t=0) = 500,000$ (gal)

$$m(t=0) = K(0) \cdot W(0) = \frac{1}{100,000} \cdot 500,000 = 5 \text{ (lb)}$$

flow of water in → $F_{w, \text{in}} = 1500$ (gal/h)

flow of water out → $F_{w, \text{out}} = 1000$ (gal/h)

$$W(t_{\text{final}}) = 600,000 \text{ (gal)}.$$

Find: $K(t_{\text{final}}) = \frac{m(t_{\text{final}})}{W(t_{\text{final}})}$

Assumptions (always hold in this lecture): Mixing of polluted & fresh waters is instantaneous; thus concentration of mercury is uniform throughout lake.

Sol'n: 1) Balance of water:

$$\frac{dw}{dt} = F_{w,in} - F_{w,out}$$

$\uparrow$ rate of change $\uparrow$ rate "in" $\uparrow$ rate "out".

$$\frac{dw}{dt} = 1500 - 1000 = 500 \Rightarrow$$

$$W = 500t + C.$$

$$\text{@ } t=0 \quad W(0) = 500,000 \Rightarrow C = 500,000.$$

$$W(t) = 500,000 + 500t. \quad (4)$$

2) Balance of amount of mercury salts:

$$\frac{dm}{dt} = F_{m,in} - F_{m,out}$$

Note: Balance equation is always written for the amount, not concentration!

$$F_{m,in} = K_{in} \cdot F_{w,in} = 0.1500 = 0$$

$$F_{m,out} = K_{out} \cdot F_{w,out} = \frac{m(t)}{W(t)} \cdot F_{w,out}$$

Thus,

$$\frac{dm}{dt} = -\frac{m}{W(t)} \cdot 1000 \quad (5a)$$

$$m(0) = 5 \quad (5b).$$

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3) Solve IVP (5). It is of the form

$$y' + p(t)y = q(t) \rightarrow 0$$

$$\frac{1000}{W(t)} = \frac{1000}{500,000 + 500t} = \frac{1}{500 + 0.5t}$$

$$y(0) = y_0 \rightarrow 5$$

and hence its solution is given by
Eq. (9) of Lecture 2:

$$P(t) = \int_0^t \frac{dt_1}{500 + 0.5t_1} \Bigg|_{t_1=0}^{t_1=t} = \int_{u=500+0.5t_1}^{u=500+0.5t} \frac{2du}{u}$$

$$= 2 \ln u \Bigg|_{t_1=0}^{t_1=t} = 2(\ln(500 + 0.5t) - \ln(500)).$$

$$\begin{aligned} m(t) \rightarrow y(t) &= y_0 e^{-P(t)} = 5 \cdot e^{-2(\ln(500+0.5t) - \ln 500)} \\ &= 5 \frac{(e^{\ln 500})^2}{(e^{\ln(500+0.5t)})^2} = 5 \cdot \left(\frac{500}{500+0.5t} \right)^2 \\ &= \frac{5}{(1+0.001 \cdot t)^2}. \end{aligned}$$

4) Find t_{final} & $m(t_{\text{final}})$.

$$\text{From (4): } W(t_{\text{final}}) = 600,000 \Rightarrow$$

$$600,000 = 500,000 + 500 \cdot t$$

$$t = 200 \text{ (hours)}$$

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$$m(t_{\text{final}}) = \frac{5}{(1 + 0.001 \cdot 200)^2} \approx 3.47 \text{ lb.}$$

$$K(t_{\text{final}}) = \frac{3.47 \text{ lb}}{600,000 \text{ gal}} \approx 5.8 \cdot 10^{-6} \frac{\text{lb}}{\text{gal}}$$