Lecture 8 An application of nonlinear DEs: the Logistic model

① Population with limited resources

For a population with unlimited resources (Lecture 4), we had

$$\frac{dP}{dt} = r_b P - r_d P$$

↑ birth rate was independent of P.

However, when members of the population begin competing for food or living space, one can write:

$$r_b = r_0 (1 - kP)$$

↑ some const > 0.

Then

$$\frac{dP}{dt} = \underbrace{r_0(1-kP)}_{r_b} P - r_d P$$

$$= (r_0 - r_d) P - r_0 k P^2 \equiv aP - bP^2.$$

↑ ↑ some positive constants.

$$= a\left(1 - \frac{b}{a} P\right) P.$$

In notations of the textbook:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{P_e}\right) P \qquad (1)$$

Note: Equilibrium solutions are for $dP/dt = 0$
$\Rightarrow P = 0$ or $1 - P/P_e = 0 \Rightarrow P = P_e$ ← "equilibrium".

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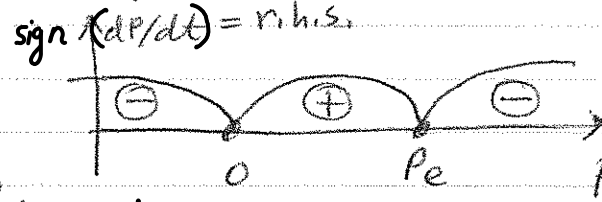
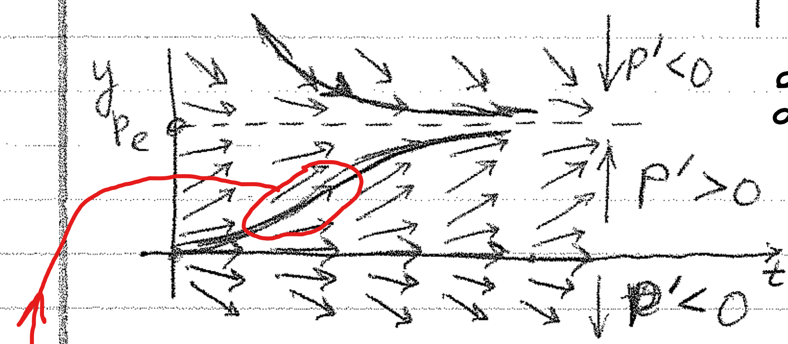
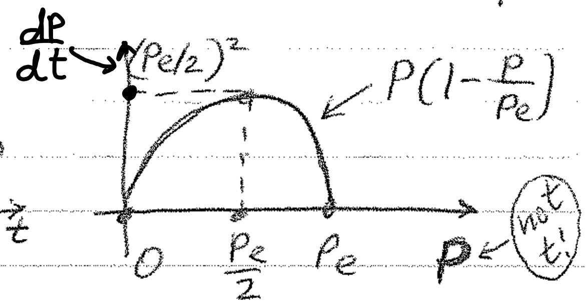
(2) Qualitative behavior of solution

$$\frac{dP}{dt} = r\left(1-\frac{P}{P_e}\right)P$$







So, as $t \to \infty$, $P(t) \to P_e$ for any $P(0) > 0$.

There is a mistake in this trajectory. Where?

(3) Exact analytical solution.

You solved the DE $\frac{dP}{dt} = P(a-P)$ at least twice in HW5/Sec. 2.6 and then also, by another method, in HW7/Sec. 2.5.

Also, the solution is presented in detail on pp. 71-72. MUST READ IT.

Result:

$$P(t) = \frac{P(0)\cdot P_e}{P(0) - (P(0)-P_e)e^{-rt}} \qquad (2)$$

Let us verify the behavior for $t \to \infty$:

$e^{-rt} \to 0 \Rightarrow$

$$P(t) \to \frac{P(0)\,P_e}{P(0)-0} = P_e.$$

✓ as predicted earlier

(4) Constant migration

$$P' = r(1-P)P + \underbrace{M}$$

↑ constant migration.

$M > 0$ — people move in
$M < 0$ — people move out

Note: A more general form would be

$$P' = r\left(1 - \frac{P}{P_*}\right)P + M$$

↑ P_*: some const., but not P_e any more!

but we will set $P_* = 1$ for simplicity.

Ex. 1 Model with migration can be transformed into one without, and hence analyzed.

Find the qualitative behavior of the solution of:

$$P' = (1-P)P \underbrace{- \frac{2}{9}}$$

$M < 0 \Rightarrow$ emigration

Sol'n:

0) Preliminaries about a linear model.

Recall $y' = ay + b$ (a, b: const) from Lecture 2.

$$y' = a(y + \underbrace{\tfrac{b}{a}}_{-c}) \Rightarrow$$

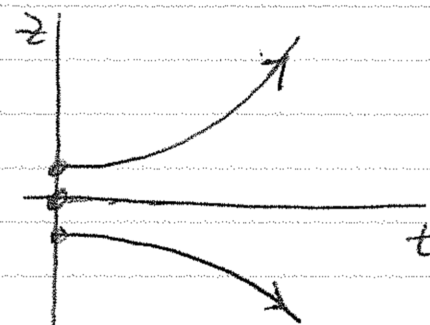
$$y' = a(y-c)$$

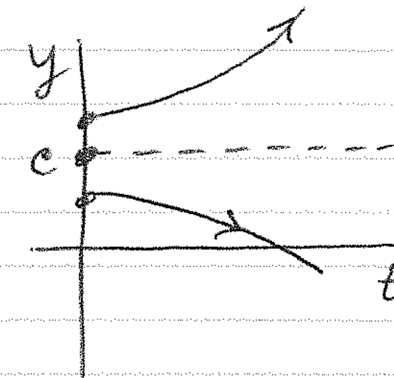
since $c' = 0$ → $$\underbrace{(y-c)}_{z}' = a\underbrace{(y-c)}_{z}$$

$$z' = az.$$

$\underline{a > 0}$

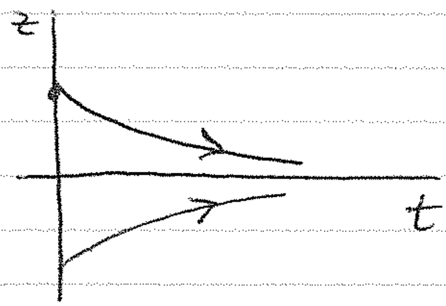
$z = Ce^{at}$

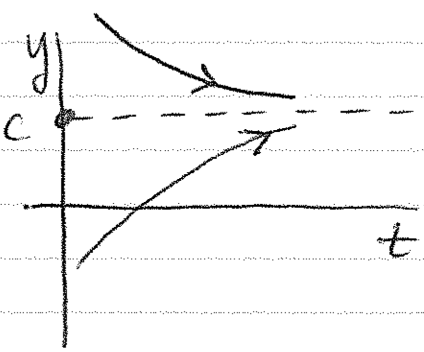




(The only) Equilibrium $y = c$ is <u>unstable</u> (solution moves away from it).

$\underline{a < 0}$





(The only) Equilibrium $y = c$ is <u>stable</u> (all solutions move <u>towards</u> it).

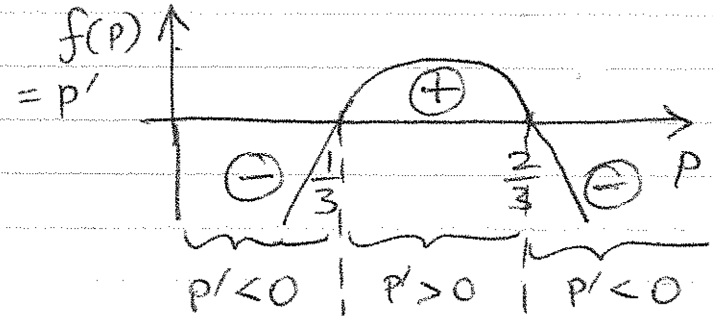
1) Come back to our <u>nonlinear</u> problem.

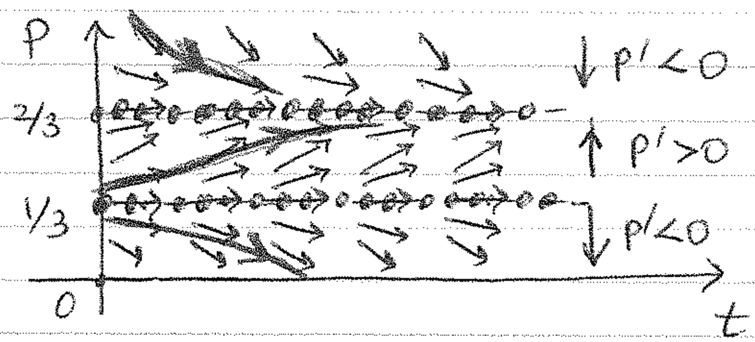
$$P' = (1-P)P - \tfrac{2}{9} = P - P^2 - \tfrac{2}{9} \equiv \underbrace{f(P)}_{\text{quadratic function}}$$

$$= -(P - \tfrac{1}{3})(P - \tfrac{2}{3})$$

(roots of $f(P)$) → $\tfrac{1}{3}$, $\tfrac{2}{3}$

Method 1 of analysis: look at (sign of $f(P)$) = (sign of P').





We find:

- there are two equilibria, $(P_e)_1 = \frac{1}{3}$, $(P_e)_2 = \frac{2}{3}$. (Multiple equilibria can occur only in nonlinear models, but not in a linear one.)
- $(P_e)_1 = \frac{1}{3}$ is the unstable equilibrium (solutions move away from it)
- $(P_e)_2 = \frac{2}{3}$ is the stable equilibrium (solutions that are sufficiently close to it ($\frac{1}{3} < P(0) < \infty$) move towards it)

Existence of stable and unstable equilibria is common for nonlinear models.

Method 2 of analysis:
Consider behavior close to $(P_e)_1$ & $(P_e)_2$

$$P' = -(P - \tfrac{1}{3})(P - \tfrac{2}{3}).$$

a) Let $P \approx (P_e)_2 = \frac{2}{3}$

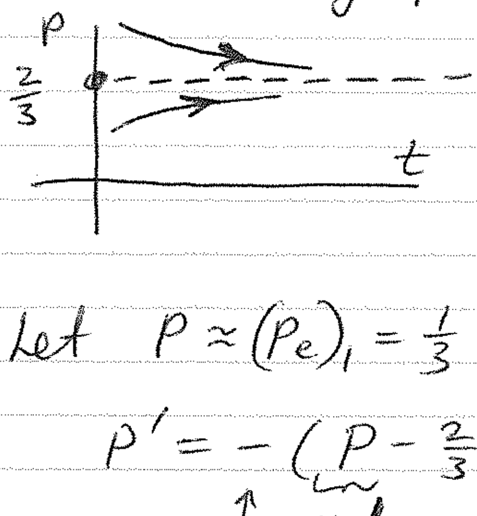
$$P' = -(\underbrace{\approx \tfrac{2}{3}}_{P} - \tfrac{1}{3}) \cdot (P - \tfrac{2}{3})$$

$\ominus \cdot \oplus \left(= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}\right)$, so $\left(-1 \cdot \frac{1}{3}\right)$

So, near $(P_e)_2 = \frac{2}{3}$, $\quad P' \approx -\frac{1}{3} \cdot (P - \frac{2}{3})$

This is Eq. (3) with $a \approx -\frac{1}{3} < 0$,

and then by p. 8-4, $P_e = \frac{2}{3}$ is stable.



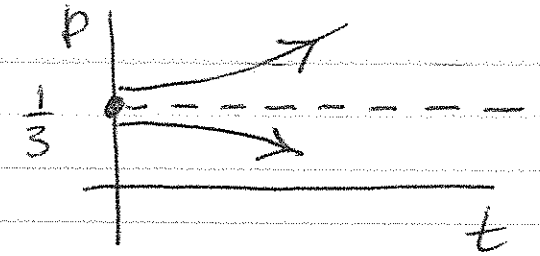
} all solutions that are sufficiently close to $P_e = 2/3$ tend towards it for $t \to \infty$.

b) Let $P \approx (P_e)_1 = \frac{1}{3}$

$$P' = -(\underbrace{P}_{\approx \frac{1}{3}} - \tfrac{2}{3})(P - \tfrac{1}{3})$$

$\ominus \quad \approx \frac{1}{3} - \frac{2}{3} \approx \left(-\frac{1}{3}\right)$, so $-1 \cdot \left(-\frac{1}{3}\right) = \frac{1}{3}$.

So, near $(P_e)_1 = 1/3$, $\quad P' \approx \left(+\frac{1}{3}\right) \cdot \left(P - \frac{1}{3}\right)$.
This is Eq. (3) with $a = +\frac{1}{3} > 0$, and then by p. 8-4, $P_e = 1/3$ is unstable.



all solutions that are sufficiently close to $P_e = 1/3$ tend **away** from it as $t \to \infty$.

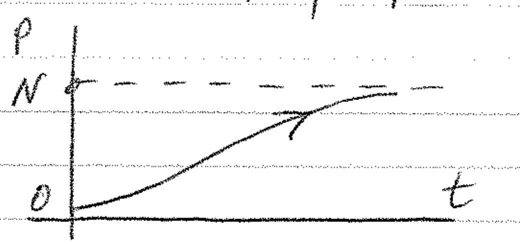
5. Other applications of the logistic model

There are many.

a) Infectious disease w/o recovery (p.74)

$$\frac{dP}{dt} = k \cdot P \cdot (N - P)$$

- N ← total
- P ← # of infected
- $N - P$ ← # of non-infected.



Everyone will get infected eventually.

b) Quadratic drag force (pp. 79-83) ← optional.