

Lecture 9Euler's method

Goal: Solve $\frac{dy}{dt} = f(t, y)$ approximately
when it cannot be solved exactly.

① Idea: Approximate dy/dt :

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h} \quad \text{for small } h.$$

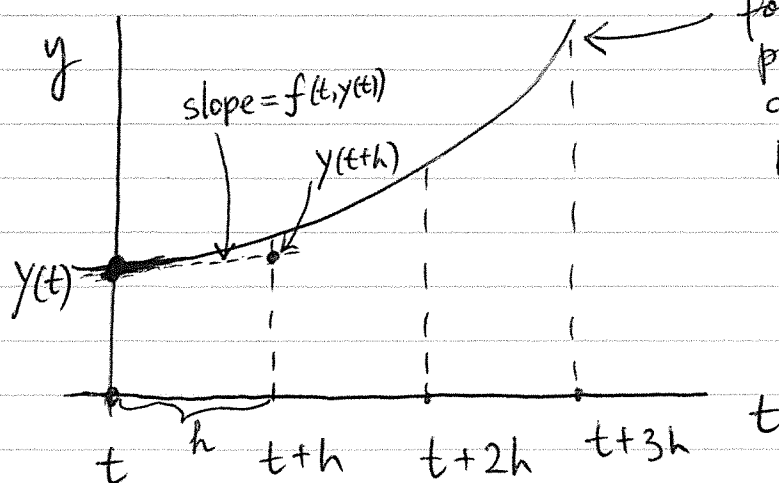
So, replace the DE with:

$$\frac{y(t+h) - y(t)}{h} = f(t, y(t))$$

and solve for the "new" value $y(t+h)$
using the known "old" value $y(t)$:

$$\boxed{y(t+h) = y(t) + h \cdot f(t, y(t))} \quad \text{Euler's Method.}$$

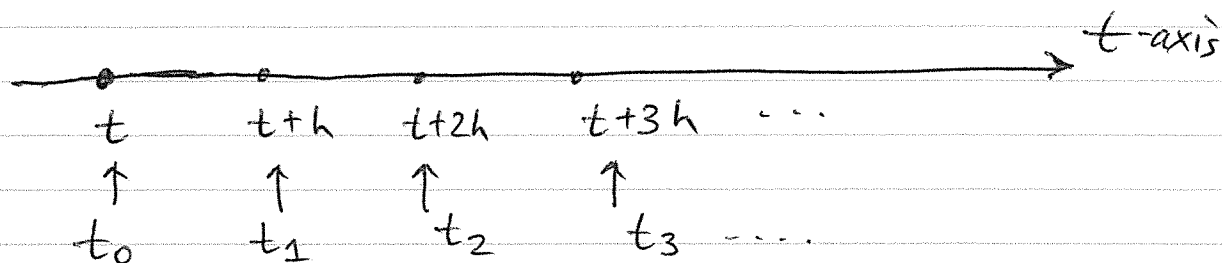
② Graphic representation



for this illustration
purpose, let's
assume that we
know the sol'n $y(t)$

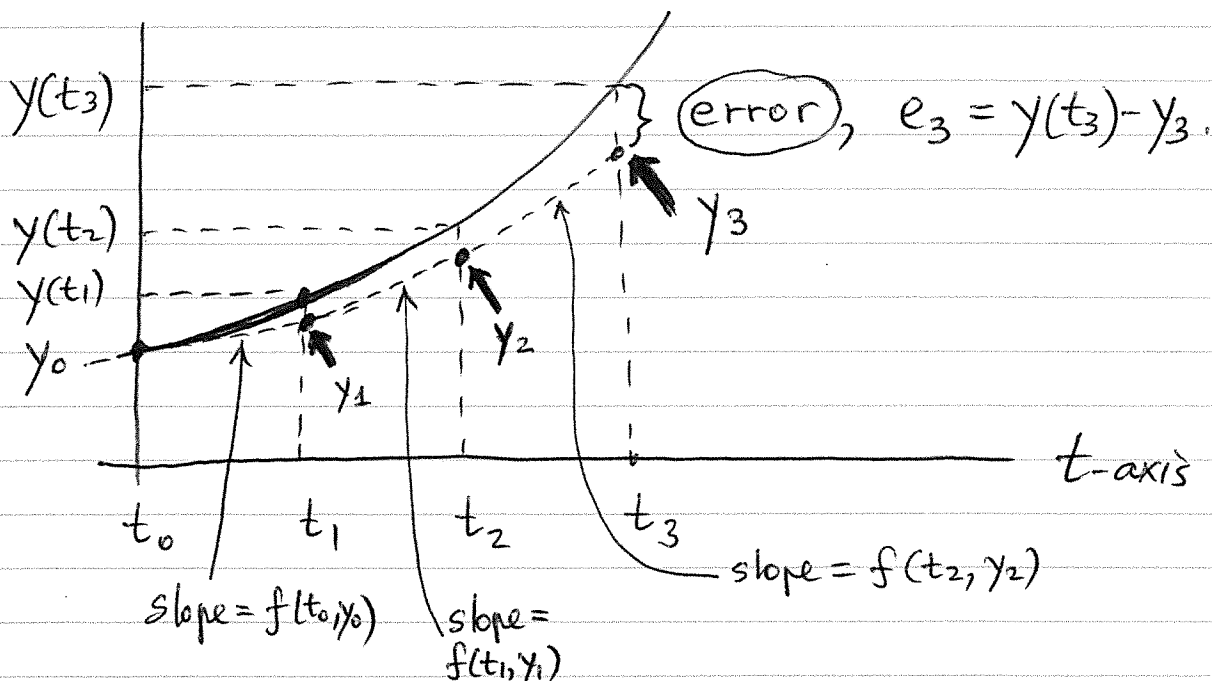
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Let's introduce a more convenient notation:



$$t_k = t_0 + k \cdot h, \quad k=0, 1, 2, \dots$$

Then we can redraw the previous figure:



Another notation: write Euler's method as

$$y_{k+1} = y_k + h \cdot f(t_k, y_k)$$

y_k = approximate solution at $t=t_k$.
 $y(t_k)$ = exact (and usually unknown) solution at $t=t_k$

Ex. 1 Compute y_1, y_2, y_3 for

$$y' = 0.5y + t, \quad y(t_0) = 3; \quad t_0 = 0;$$

while using $h = 0.1$.

Sol'n:

$$y_{k+1} = y_k + h \cdot f(t_k, y_k)$$

$$= 3 \Rightarrow k = 2.$$

stop at $k = 2$
(for y_3)

Decide on the k
where
you
need to
stop.

Thus need to do the calculations for
 $k = 0, 1, 2$.

$k = 0$

$$\begin{aligned} y_{0+1} &= y_0 + h \cdot f(t_0, y_0) \\ &= 3 + 0.1 \cdot (0.5 \cdot y_0 + t_0) \\ &= 3 + 0.1 \cdot (0.5 \cdot 3 + 0) = 3.15 \end{aligned}$$

$\downarrow y_1$

$k = 1$

$$\begin{aligned} y_{1+1} &= y_1 + h \cdot f(t_1, y_1) \\ &= 3.15 + 0.1 \cdot (0.5 \cdot y_1 + t_1) \\ &= 3.15 + 0.1 \cdot 1.675 = 3.3175 \end{aligned}$$

$\downarrow y_2$

$k = 2$

$$\begin{aligned} y_{2+1} &= y_2 + h \cdot f(t_2, y_2) \\ &= 3.3175 + 0.1 \cdot (0.5 \cdot y_2 + t_2) \\ &= 3.3175 + 0.1 \cdot 1.85875 = 3.503375 \end{aligned}$$

$\downarrow y_3$

Exact sol'n (Lec. 2): $y(t_3) = 3.53284$.

$\uparrow$
(0.3)

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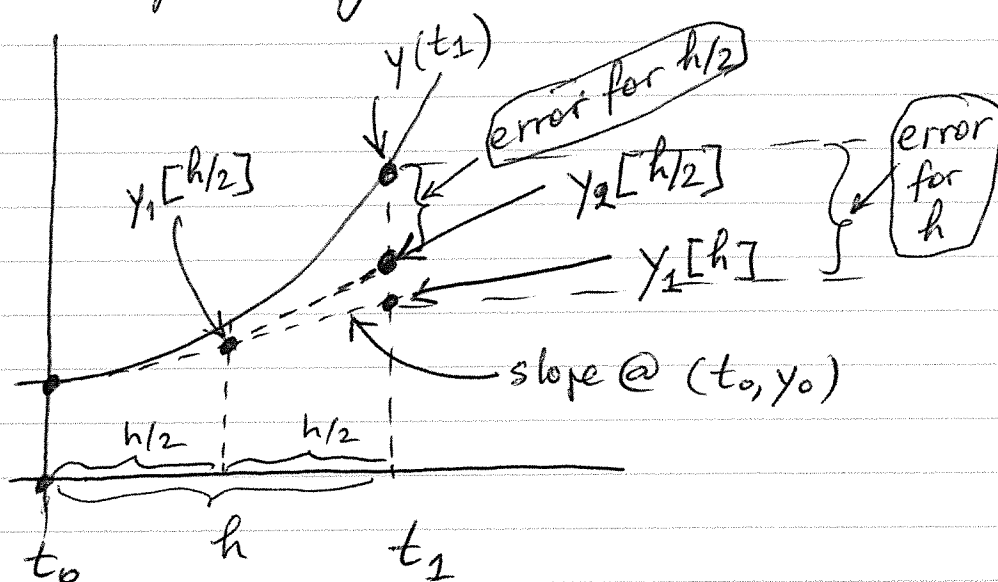
The error at $t_3 = 0.3$ is

$$|y(t_3) - y_3| = |3.53284 - 3.503375| \approx 0.0295,$$

or $\approx 0.9\%$.

Note: The error will decrease if we take a smaller h (expectedly)!

Why? Graphically:



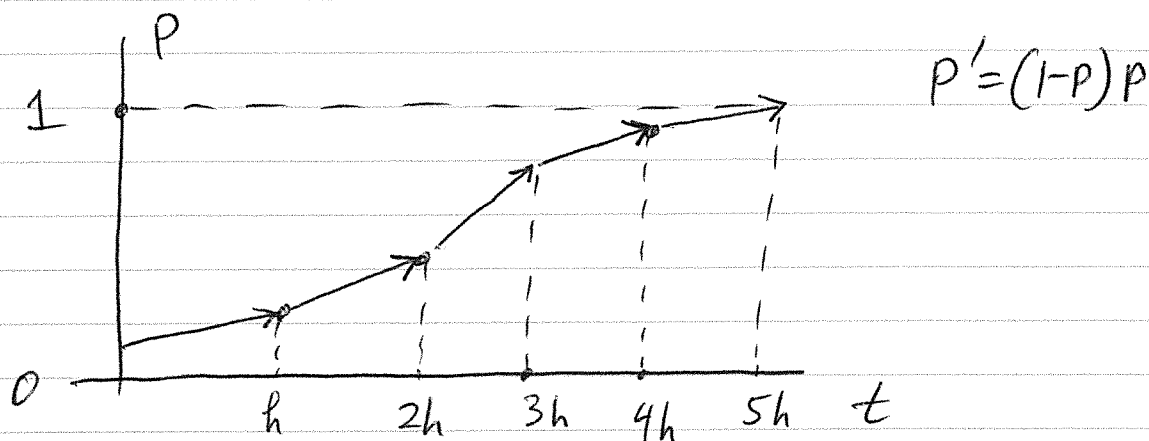
We see that $y_2[h/2]$ is a better approximation to $y(t_1)$ than $y_1[h]$.

Mnemonically: The more accurate solution $y_{2k}[h/2]$ is

always found between the exact sol'n $y(t_k)$ and the less accurate sol'n $y_k[h]$.

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③ Relation to direction field:



The approximate solution found by Euler's method follows arrows of the direction field.