MATH 230

Elementary Differential Equations

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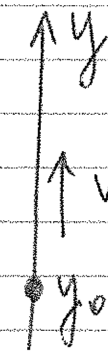
Lecture 1. Introduction

1 Motivation

Ex. 1 a) An object starts at y_0 and moves along y-axis with speed v.



Find location $y(t)$ at any t.

Sol'n: $y(t) = y_0 + vt$.

b) Q: What if $v \neq$ const?

We know: $\boxed{y'(t) = v(t)}$ (1) ← integrate

$$y(t) = \int v(t)\,dt + C \qquad (2)$$

↑ explicitly account for arb. constant.

Constant C is found from the initial condition.

However, we now use an integration method which will find C right away. (We'll use the same method many times in this course.)

$$\int_0^t y'(t_1)\,dt_1 = \int_0^t v(t_1)\,dt_1$$

we will explain this notation later

by Fund. Thm. of Calculus $= y(t) - y(0)$.

~~so~~:

$$y(t) - y(0) = \int_0^t v(t_1)\,dt_1$$

$$\boxed{y(t) = y_0 + \int_0^t v(t_1)\,dt_1} \qquad (3)$$

Moral: Our goal is:

Given the diff eqn. $\frac{dy}{dt} = f(t,y)$ (4a)

and initial condition $y(t_0) = y_0$ (4b)

We want to find solution $y(t)$.

Note 1 The combination of diff. eqn. (DE), (4a), and initial condition, (4b), is called initial value problem (IVP).

Note 2 When we solve any DE, we get an arbitrary constant, C (see (2)). Its value is found from the initial cond.; e.g., in (3), $C = y_0$.

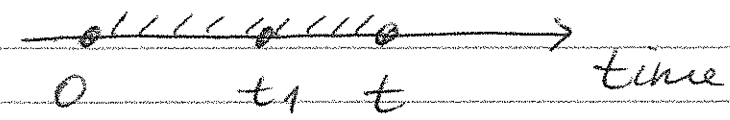
Note 3 (about notations)

When we integrate $v(t)$ from 0 to t, we write:

$$\int_0^t v(t_1)\,dt_1$$

this is the time up to which we integrate

This is a "dummy" variable ($0 \le t_1 \le t$).



It must have a different name than the finite integration time.

Sometimes, we may use $\tilde{t}$ (t-tilde) instead of t_1:

$$\int_0^t v(\tilde{t})\,d\tilde{t}.$$

Note 4 DE (4a) is more complicated than DE (1). Indeed, if we write its solution as:

$$y(t) = \int f(t, y(t))\,dt + C, \qquad (5)$$

we see that we cannot integrate because we do not know the unknown solution $y(t)$ inside $f(t, y)$.

So we need to develop methods of how to find $y(t)$ other than by (5).

(2) Examples of DEs

Ex. 2 Given IVP: (a = const)

$$y' = ay \qquad (6)$$

$$y(0) = 3$$

a) Show that the solution of this DE is

$$y = Ce^{at} \quad (C = \text{const})$$

b) Find the solution of the IVP.

Sol'n: a) We are asked to verify a given solution, so we simply compare the lhs. and rhs.

Chain Rule

lhs: $y' = (Ce^{at})' = C(e^{at})' = C \cdot (at)' \cdot e^{at}$

$= C \cdot a \cdot e^{at}$

rhs: $a \cdot y = a \cdot Ce^{at}$.

lhs = rhs. ✓

b) $Ce^{at}\Big|_{t=0} = 3 \Rightarrow C \cdot \cancel{e^0}^{\,1} = 3 \Rightarrow C = 3$.

$$y(t) = 3e^{at}.$$

Note: DE (6) is the most basic DE which arises in many applications in all branches of science (physics, chemistry, ...)

It basically says:

$$\underbrace{y'}_{\text{rate of change of } y} \; \underbrace{= a \cdot}_{\text{is proportional to}} \; \underbrace{y}_{\text{instantaneous value of } y.}$$

For example:

sub- Ex. 2(a)

$$\frac{dm}{dt} = -a \cdot m$$

mass of radioactive material

(radioactive decay law: a given percentage of material decays in a unit of time).

sub- Ex. 2(b) ← Newton's Cooling Law

$$\frac{d(T-T_r)}{dt} = -k \cdot \underbrace{(T-T_r)}_{\text{temp. of coffee in excess of room temperature}}$$

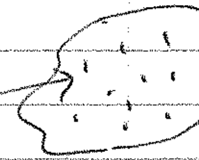
T

coffee

T_r = room temp.

temperature of coffee

sub- Ex. 2(c)



$$\frac{dn}{dt} = b \cdot n$$

number of bacteria multiplying in presence of unlimited supply of food.

3 Some terminology

1) $y'(t) = f(t, y(t))$

$t \leftarrow$ independent variable

$y \leftarrow$ dependent variable

2) Ordinary DE : $\frac{dy}{dt}$ is ordinary derivative.

will NOT study in math 230

Partial DE: $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$

involves partial derivatives of dependent variable $u(x,t)$ w.r.t. two independent variables x, t.

3) Order of DE = order of highest derivative

Ex. 3(a)

$$y'' = 1$$

$\uparrow$ 2nd der., $\Rightarrow$ order $= 2$

Ex. 3(b)

$$\sin(y') + \cos(y''') = 0$$

$\uparrow$ 3rd der., $\Rightarrow$ order $= 3$

Ex. 3(c)

$$(y'')^5 + (y')^2 + \ln y = 0$$

$\uparrow$ 2nd der., $\Rightarrow$ order $= 2$.

4) Autonomous & non-autonomous DE.

$y' = f(y)$ ← autonomous.

$y' = y^2 - \sin(e^y)$.

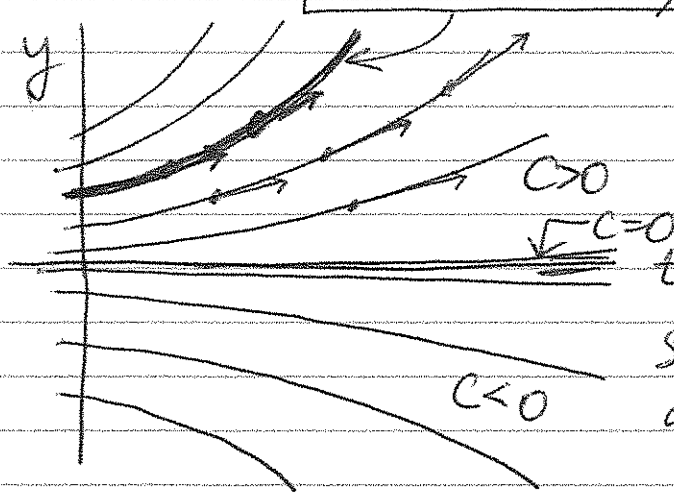
$y' = f(t, y)$ ← non-autonomous

$y' = y^2 - t$

4 Direction fields

Idea: Recall Ex. 2.

IVP has 1 solution $y(t)$.



But DE has ∞ many solutions because of arb. const C.

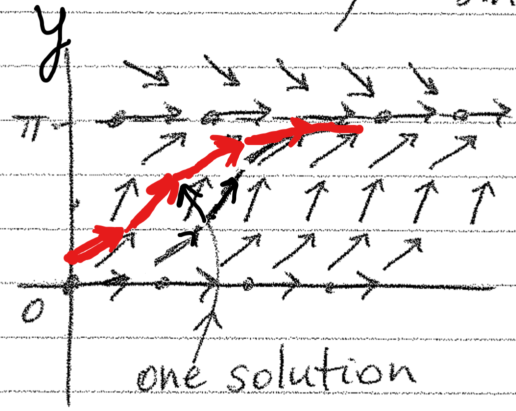
So sol'ns of DE are many curves.

Direction field is the collection of all these curves with arrows at each point indicating the direction of motion.

See also a more detailed version of this example posted next to this Lecture.

Ex. 4 Sketch the direction field of

$$y' = \sin y$$



Sol'n:

This is an autonomous DE, ⇒ y' is the same along each line $y = \text{const}$.

$\underline{y=0} \Rightarrow \sin y = 0 \Rightarrow y' = 0$

$\underline{y = \frac{\pi}{4}} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \simeq 0.7, \quad y' \simeq 0.7$

$y = \frac{\pi}{2} \Rightarrow \sin y = 1, \quad y' = 1$

$y = \frac{3\pi}{4} \Rightarrow \sin y = \frac{1}{\sqrt{2}} \simeq 0.7, \quad y' = 0.7$

$y = \pi \quad \sin y = 0, \quad y' = 0$

$y = \frac{5\pi}{4} \quad \sin y = -\frac{1}{\sqrt{2}} = -0.7, \quad y' = -0.7$

Equilibrium solution:

A constant solution such that $y' = 0$ for all t:

$$y' = f(t, y_0) = 0.$$

That is, $y(t) = y_0$ is a sol'n for all t.

<u>Ex. 5 (a)</u> In Ex. 4, equilibrium sol'ns are $y = 0, \pi, 2\pi$, etc.

<u>Ex. 5 (b)</u> In
$$y' = t(y+1)^2,$$
the equilibrium sol'n is $y = -1$.

<u>Ex. 5 (c)</u> In
$$y' = ty + 1,$$
there are <u>no</u> equilibrium solutions,

because $y' = 0$ implies $y = -1/t$, which contradicts $y' = 0$ $\left((-1/t)' = 1/t^2 \neq 0\right)$.

<u>Ex. 5 (d)</u> In
$$y' = y^2 + 1$$
there are also <u>no</u> equilib. solutions because $(y^2+1) > 0$ for all y.

<u>HW</u>: <u>Sec. 1.2</u> ## 5, 7, 8, 9, 13, 14, 21, 23
Hint for #23: Find a solution of this problem in a Calc. textbook (usually in Calc. III, under „Velocity & acceleration").
\# 3
\# 10 (Hint: proceed similarly to #23)

<u>Sec. 1.3</u> 5, 1, 2, 4, 10.