Lecture 10. Introduction to 2nd-order DEs: motivation, basic properties, linear DEs.

① Definition

$$y'' = F(t, y, y') \qquad (1)$$

is called a 2nd-order DE (shorthand notation: DE-2, as opposed to the 1st-order DEs, or DE-1).

If F is a nonlinear function of y and/or y', then DE-2 can be solved analytically (i.e. not numerically by a computer) only in very few special cases (even much more rarely than a nonlinear DE-1). E.g., $y'' = f(y)$ (an autonomous eq.) cannot be solved except for some very special $f(y)$.

Then we look at the linear DE-2:

We'll study only this DE in Chap. 3

$$y'' + p(t)\,y' + q(t)\,y = g(t). \qquad (2)$$

Unlike linear DE-1, Eq. (2) cannot be solved for general $p(t)$ & $q(t)$. This is a big difference between linear DE-1 & DE-2. We'll show it later in this lecture.
In general, solving a DE-2 is much harder than a DE-1.

However, there are also some similarities in their properties.

2 Motivation

Q: Why consider DE-2?

A: Most processes in Mechanics are governed by the 2nd Law of Newton:

$$\underbrace{m\vec{a}}_{\text{acceleration}} = \underbrace{\sum \vec{F}}_{\text{vector sum of all forces}}$$

In one dimension, $\vec{a} = \dfrac{d^2y}{dt^2}$, $\Rightarrow$

$$m y'' = F(t, \underbrace{y}_{\text{position}}, \underbrace{y'}_{\text{velocity}}) \qquad (3)$$

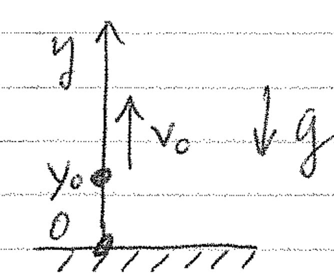
which has the form (1).

3 What to expect of the solution?

Consider a very simple DE-2 (HS Physics or Calc. III):

$$m y'' = -\underbrace{mg}_{\text{gravity}}$$

$$\begin{cases} y'' = -g \\ y(0) = y_0, \ y'(0) = v_0 . \end{cases}$$



Solution: 1) $y'' = -g \Rightarrow y' = -\int g\,dt = -gt + C_1$

$y'(0) = v_0 = -g\cdot 0 + C_1 \Rightarrow C_1 = v_0 .$

2) $y' = -gt + v_0 \Rightarrow y = \int(-gt + v_0)\,dt$
$= -gt^2/2 + v_0 t + C_2$.

$y(0) = y_0 = -g\cdot 0 + v_0 \cdot 0 + C_2 \Rightarrow C_2 = y_0$.

Thus $y(t) = -\frac{gt^2}{2} + v_0 t + y_0$

Observations: 1) We have two constants, C_1 & C_2, not one as for DE-1.

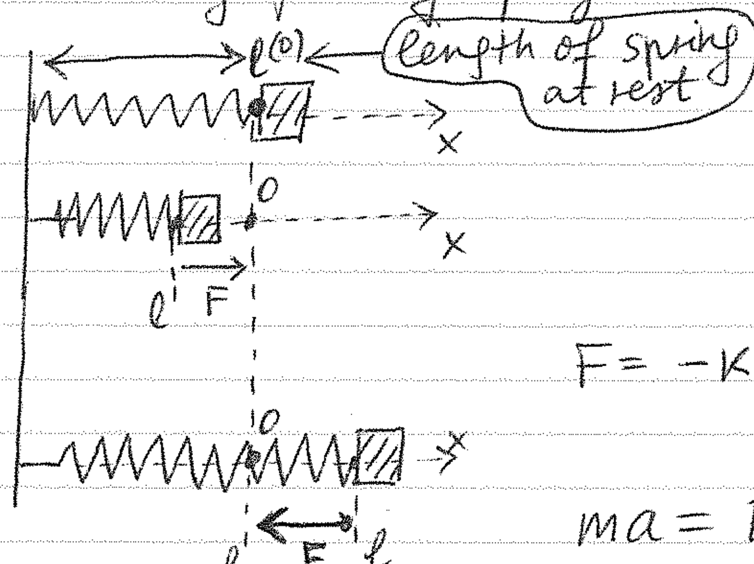
2) We need two initial cond's, $y(0)$ & $y'(0)$, to determine these constants (although, in general, they depend on $y(0)$, $y'(0)$ in a more complicated way).

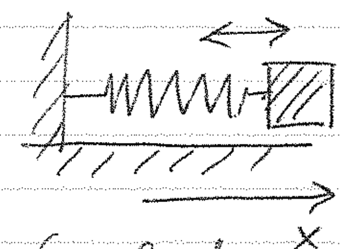
Meanings: $y(0)$ = initial position; $y'(0)$ = initial velocity.

4 A fundamental model: linear oscillator

Consider the motion of a mass on a spring:

1) Restoring force of spring





(neglect friction)

$F = -k\underbrace{(\ell - \ell_0)}_{x}$;

$ma = F \Rightarrow$

$$mx'' = -kx$$

$$mx'' + kx = 0$$

$$x'' + \frac{k}{m}x = 0$$

$\frac{k}{m} \to \omega^2$ ← "omega square"

$$\boxed{x'' + \omega^2 x = 0} \quad (4)$$

Since mass on a spring oscillates, Eq. (4) is called the <u>linear</u> <u>oscillator model</u>.

It is fundamental to DE-2s, just like $y' = ay$ is fundamental for DE-1.

2) <u>General sol'n</u> (to be derived later, in Sec. 3.5)

a) $x_1 = \cos\omega t$ is a sol'n of (4).

<u>Check:</u>

$$x_1' = -\omega\sin\omega t$$

$$x_1'' = -\omega\cdot\omega\cdot\cos\omega t = \underline{-\omega^2\cos\omega t}.$$

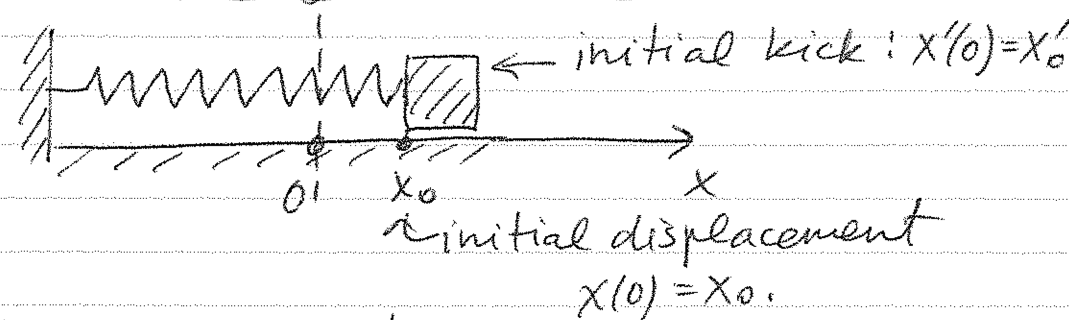
$$x_1'' + \omega^2 x_1 = -\omega^2\cos\omega t + \omega^2\cdot\cos\omega t = 0. \quad \checkmark.$$

b) $x_2 = \sin\omega t$ is also a sol'n of (4)
(proof: @ home)

c) $x = c_1x_1 + c_2x_2 = c_1\cos\omega t + c_2\sin\omega t$; $c_{1,2} = \text{const}$
is also a solution of (4) (proof: @ home).

<u>Note:</u> Looks like the superposition principle for DE-1.

3) Solution for given init. cond's.



Then:

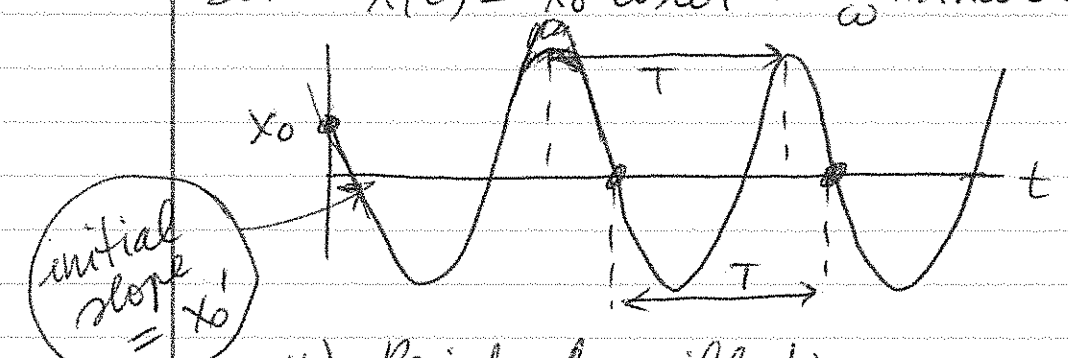
$$X(0) = C_1\cdot\underbrace{\cos(\omega\cdot 0)}_{1} + C_2\cdot\underbrace{\sin(\omega\cdot 0)}_{0} = X_0$$

$$X'(0) = C_1(\cos\omega t)'\Big|_{t=0} + C_2(\sin\omega t)'\Big|_{t=0}$$

$$= -C_1\cdot\omega\cdot\underbrace{\sin(\omega\cdot 0)}_{0} + C_2\cdot\omega\cdot\underbrace{\cos(\omega\cdot 0)}_{1}$$

$$\Rightarrow \begin{array}{l} C_1\cdot 1 + C_2\cdot 0 = X_0 \\ C_1\cdot 0 + C_2\cdot\omega = X_0' \end{array} \Rightarrow \begin{array}{l} C_1 = X_0 \\ C_2 = X_0'/\omega. \end{array}$$

So: $X(t) = X_0\cdot\cos\omega t + \frac{X_0'}{\omega}\cdot\sin\omega t.$



4) Period of oscillations

Note: $\boxed{\cos(\omega t_1 + 2\pi) = \cos\omega t_1} \equiv \cos\left(\omega\left[t_1 + \frac{2\pi}{\omega}\right]\right)$ (due to 2π-periodicity)

Similarly, $\sin(\omega t_1 + 2\pi) = \sin\omega t_1$.

Thus the solution exactly repeats itself

We will see examples of this later.

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after $t_2 - t_1 = 2\pi/\omega$. So $T = 2\pi/\omega$ is called the period of the oscillations.

$$\boxed{\begin{array}{ll} \omega = \text{frequency}, & T = \text{period} \\ \omega = \dfrac{2\pi}{T} & T = \dfrac{2\pi}{\omega} \end{array}} \qquad (5)$$

5 Existence & uniqueness of solution of a linear DE-2

Note that this Thm is about the solution of the IVP, not of the DE! If the condition of the Thm. is violated, the solutions of the DE may still exist, but we will not be able to solve the IVP.

Thm. 3.1

Let $p(t), q(t), g(t)$ be continuous on $t \in (a,b)$ and let $t_0 \in (a,b)$. Then the sol'n of IVP

$$y'' + p(t)y' + q(t)y = g(t),$$

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

exists & is unique for $t \in (a,b)$.

Note: This statement is exactly the same as for a DE-1 (Sec. 2.1), except that here we also have an extra coeff. $q(t)$.

6 Linear DE-2 cannot be solved analytically for general $p(t)$ & $q(t)$.

Note 1 This is in stark contrast to the situation with linear DE-1, which can be solved for any $p(t), g(t)$ (see Lec. 2).

Note 2 We'll demonstrate our statement for a homogeneous linear DE-2 ($g(t)=0$).

Later we'll show that if a homogeneous linear DE-2 can be solved, then so can be its non-homogeneous version, too.

We are going to show that a linear DE-2 is equivalent to the Riccati DE-1, which cannot be solved in general (Lec. 7).

Consider

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad (6)$$

and a change of variables:

$$y = e^{-\underbrace{\int v(t)\,dt}_{V(t)}}, \qquad (7)$$

where $v(t)$ is the new variable.
To substitute (7) into (6), we need y' & y''.

a) $y' \underset{\text{Chain R.}}{=} (-V)' \cdot e^{-V} = -v \cdot e^{-V}$

b) $y'' = (y')' = (-v \cdot e^{-V})' \overset{\text{product R.}}{=} -v' e^{-V} - v \cdot (e^{-V})'$

$= -v' e^{-V} - v \cdot (-v e^{-V}) = -v' e^{-V} + v^2 e^{-V}$.

Substitute these into (6):

$$\underbrace{(-v'\cancel{e^{-V}} + v^2 \cancel{e^{-V}})}_{y''} + p(t)\underbrace{(-v \cdot \cancel{e^{-V}})}_{y'} + q(t) \cdot \underbrace{\cancel{e^{-V}}}_{y} = 0$$

(10-8)

$$-v' + v^2 - p(t)v + q(t) = 0$$

$$v' + p(t)v = v^2 + q(t) \leftarrow \text{Riccati for } v.$$

Since one cannot find v for general p & q, so one cannot find y from (7) for general p & q.

$\leftarrow$ Section 3.1

HW: 3.1.11 $\leftarrow$ graph, concavity

9, 10 $\leftarrow$ sol'n of oscillator equ. (Ans. for #10: $C_1 = 1$, $C_2 = -1$.)

13 $\leftarrow$ freq. of a bobbing object (see Eq. (3) in book).

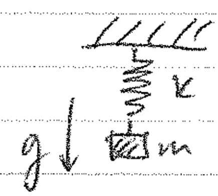
14 $\leftarrow$ parameters of osc. model from graph

Ans.: (a) $y_0 = 0$, $T = 2$, (b) $\omega = \pi$, $y_0' = 6\pi$.

3, 7 $\leftarrow$ existence/uniqueness interval

WP 1: (a) verify that $x = \sin \omega t$ is a sol. of lin. oscill.
(b) same for $x = c_1 \cos \omega t + c_2 \sin \omega t$.

WP 2:



$y'' = -\omega^2 y - g.$
Use trick of top p. 5, Lec. 2 to find the general sol'n.