

Lecture 11. The general solution of a linear homogeneous 2nd-order DE.

We will consider

$$y'' + p(t)y' + q(t)y = 0 \quad (1)$$

$p(t), q(t)$ are continuous for $t \in (a, b)$.

Then, by Thm. 3.1, $y(t)$ exists & is unique on (a, b) .

① Superposition principle

Thm. 3.2 Let $y_1(t), y_2(t)$ be two solutions of (1) and c_1, c_2 be any constants. Then:

$$y_3(t) = c_1 y_1(t) + c_2 y_2(t)$$

is also a sol'n of (1).

Note 1: This is exactly the same as the Superposition Principle (part (a)) for DE-1 (see Lec. 3), and the proof is exactly the same. See also p. 116 in textbook.

Note 2: Later we'll consider the Superposition Principle (part (b)) for non-homogeneous DE-2 and will show that it is also the same as for DE-1.

2/22/16

② Fundamental sets of solutions

2a Motivation and definition

Thm. 3.2 said:

$$\left(\begin{array}{l} y_1, y_2 \text{ are} \\ \text{sol'n's of (1)} \end{array} \right) \Rightarrow \left(\begin{array}{l} y_3 = c_1 y_1 + c_2 y_2 \\ \text{is also a sol'n of (1)} \end{array} \right).$$

Q: Are there two sol'n's $y_1(t), y_2(t)$ of (1) such that any sol'n $y(t)$ of (1) can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some suitable constants c_1, c_2 ?

Def: Such two solutions y_1 & y_2 , if they exist, are called the fundamental set of solutions of (1).

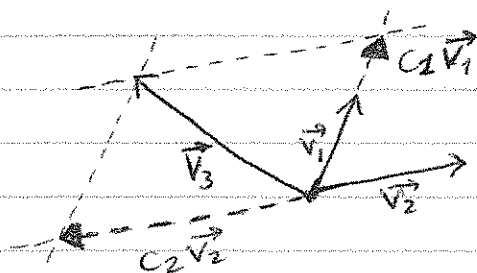
Analogy with vectors in the plane:

Q: Are there two vectors $\vec{v}_1$ & $\vec{v}_2$ such that any vector $\vec{v}_3$ in the same plane can be written as

$$\vec{v}_3 = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

for some c_1, c_2 ?

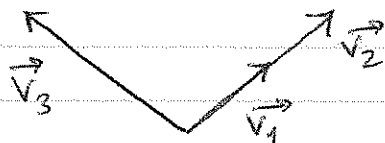
A: Any two non-parallel vectors $\vec{v}_1$ & $\vec{v}_2$ will do.



Thus, any two non-parallel $\vec{v}_1$ & $\vec{v}_2$ form a fundamental set of vectors in the plane.

11-3

Note: Two parallel vectors will not form a fundamental set:



$$\vec{v}_3 \neq c_1 \vec{v}_1 + c_2 \vec{v}_2 \text{ for any } c_1, c_2.$$

in principle,
not analytically

Observations:

1) It is quite likely that one can find a fundamental set of solutions of a DE, just as we've found a fundamental set of vectors in the plane.

2) We can think of solutions of a homogeneous DE as vectors!

Using the definition of a vector space (from Linear Algebra), one can show that they are equivalent. (But we will not show this.)

For a Example of finding c_1, c_2 for solutions of a DE-2 instead of vectors

MUST SEE
EX. 1 in SEC. 3.2

26 What solutions can form a fundamental set?
The Wronskian.

Problem: let $y_1(t), y_2(t)$ be solutions of

$$y'' + p(t)y' + q(t)y = 0. \quad (1)$$

Find $c_1, c_2 = \text{const.}$ such that

$$y = c_1 y_1 + c_2 y_2 \quad (2)$$

(11-4)

satisfies the initial condition

$$y(t_0) = y_0, \quad y'(t_0) = y'_0. \quad (3)$$

(Recall that we know that by Thm. 3.2, this solution exists and is unique.)

Sol'n: Substituting (2) into (3) we find:

$$\begin{aligned} y(t_0) &= y_1(t_0)c_1 + y_2(t_0)c_2 = y_0 \\ y'(t_0) &= y'_1(t_0)c_1 + y'_2(t_0)c_2 = y'_0 \end{aligned} \quad (4)$$

Recall matrix multiplication:

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1x_1 + a_2x_2 \\ b_1x_1 + b_2x_2 \end{pmatrix}$$

Then eqs. (4) can be written as one matrix eq.:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} \quad (5a)$$

$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$
 $A \qquad \qquad \vec{x} \qquad \qquad \vec{b}$

$$A \vec{x} = \vec{b}. \quad (5b)$$

On one hand, we know from Thm. 3.1 that $y(t)$, which is the solution of the IVP (1) & (3), exists and is unique.

On the other hand, from Linear Algebra we know that the solution $\vec{x}$ of (5b) exists and is unique iff

$$\det(A) \neq 0, \text{ i.e.}$$

$$\rightarrow W(t_0) \equiv \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0 \quad (6)$$

"defined as"

New notation: Wronskian of y_1 & y_2 .

Thus, so far we have:

$$\left(\begin{array}{l} (y_1(t), y_2(t)) \text{ is a} \\ \text{fundamental set} \end{array} \right) \Rightarrow (W(t_0) \neq 0). \quad (7a)$$

Conversely, from Lin. Algebra we know that if $\det A \neq 0$ in (5b), then its solution $\vec{x}$ exists for any r.h.s. vector $\vec{b}$.

Then, equivalently, if $W(t_0) \neq 0$ in (5a), then a unique solution $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ always

exists for any $\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$. Since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ exists,

then by (2), the solution of the IVP (1)&(3) can be written as $c_1 y_1 + c_2 y_2$.

Thus:

$$(W(t_0) \neq 0) \Rightarrow (y_1, y_2 \text{ is a fundamental set}). \quad (7b)$$

11-6

To combine (7a) & (7b):

$$\left(y_1(t), y_2(t) \text{ is a fundamental set} \right) \Leftrightarrow \left(W(t_0) \neq 0 \right). \quad (7c)$$

Note: In a later lecture we'll show that if (a, b) is the interval mentioned in the Existence & Uniqueness Thm for the DE satisfied by y_1 & y_2 , then

$$\text{either } \left(W(t) \neq 0 \text{ for all } t \in (a, b) \right) \text{ or } \left(W(t) = 0 \text{ for all } t \in (a, b) \right)$$

In other words, $W(t)$ cannot cross 0 in the interval where the coeff's $p(t), q(t)$ of the DE are continuous. (Later we will see that y_1, y_2 can exist and be continuous even when $p(t), q(t)$ are not. So above we stress the continuity of p, q and not of y_1, y_2 .)

Ex. 1(a) Show that $y_1 = \cos \omega t, y_2 = \sin \omega t$ (see Lec. 10, topic (4)) form a fundamental set of solutions of the linear oscillator model:

$$y'' + \omega^2 y = 0 \quad (8)$$

Note: $p(t) = 0, q(t) = \omega^2 = \text{const}$ are continuous everywhere, $\Rightarrow y(t)$ exists for all t .

According to the Note above, we just need to compute $W(t_0)$ for some t_0 . Instead of name " t_0 ", just use " t ".

$$\begin{aligned} \text{Sol'n: } W(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{vmatrix} \\ &= \omega \cdot \cos^2 \omega t - (-\omega \sin^2 \omega t) = \omega (\cos^2 \omega t + \sin^2 \omega t) = \omega \neq 0. \end{aligned}$$

Thus, by (7c), $\{\cos \omega t, \sin \omega t\}$ forms a fundamental set.

Ex. 1(b) show that $y_1 = \cos \omega t$, $y_2 = A \cos \omega t$ ($A = \text{const}$) is not a fundamental set of (8).

Sol'n: $W(t) = \begin{vmatrix} \cos \omega t & A \cos \omega t \\ -\omega \sin \omega t & A(-\omega \sin \omega t) \end{vmatrix}$

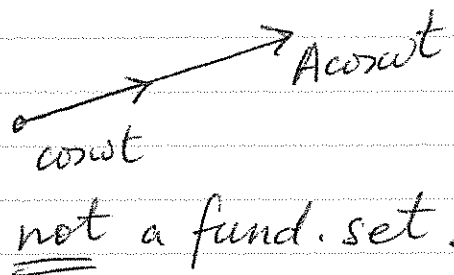
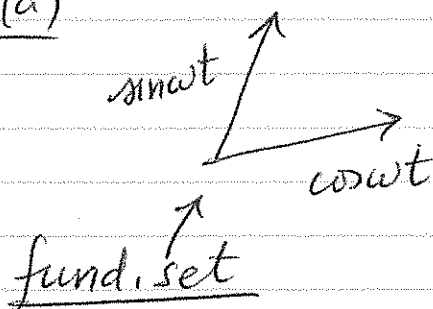
$= A \cdot (-\omega \cdot \cos \omega t \cdot \sin \omega t) - A \cos \omega t \cdot (-\omega \sin \omega t) = 0.$

Again, by (7c), $\{\cos \omega t, A \cos \omega t\}$ is not a fund. set.

See also Ex. 3 in Sec. 3.2.

Analogy with vectors:

1(a)



HW. Sec. 3.2 $\begin{cases} 1, 3, 6 \text{ (} b \rightarrow W = \frac{1}{2}e^{t/2}, c \rightarrow \text{Use DSolve)} \\ 7, \\ 9, 10 \text{ (} b \rightarrow W = -\ln^3 t, c \rightarrow \text{Use DSolve)} \\ 12, 13 \end{cases}$

Find W and then C_1, C_2 .

W & fund. set $\rightarrow \begin{cases} 21 \text{ (Hint: Use Eqs. (7) of Notes.)} \\ 20 \text{ (} \alpha \neq \pi n \end{cases}$

similar to $\begin{cases} 17, 16 \text{ (Hint: See Ex. 1 of Sec. 3.2)} \\ (b \rightarrow C_1 = 1/(2\sqrt{2}), C_2 = 1/\sqrt{2}). \end{cases}$

WP: In Ex. 3 for Sec. 3.2, $W(t)$ crosses zero (at $t=0$). Why does this not contradict the first Note on p. 11-6?