

Lecture 14. Complex roots; oscillatory solutions of DE-2

① Meaning of the complex exponential.

1a In Case 3 listed in Lecture 12,

$$\lambda_{1,2} = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a} \\ \equiv \alpha \pm i\beta, \quad (1)$$

if $(4ac - b^2 > 0)$ in the characteristic eq.

$$a\lambda^2 + b\lambda + c = 0 \quad (2)$$

Then the DE-2:

$$ay'' + by' + cy = 0 \quad (3)$$

has two solutions

$$y_1 = e^{(\alpha+i\beta)t}, \quad y_2 = e^{(\alpha-i\beta)t}. \quad (4)$$

Moreover, since $\lambda_1 \neq \lambda_2$, then as shown in Lecture 12 and p. 124 of book,

3/4/16 $\{y_1, y_2\}$ is a FS.

Ex. 1 Suppose $a=1, b=0, c=\omega^2$.

Then (2) becomes

14-2

$$\lambda^2 + \omega^2 = 0 \Rightarrow$$

$$\lambda^2 = -\omega^2 \Rightarrow \lambda = \pm i\omega. \quad (5)$$

Then the corresponding DE

$$y'' + \omega^2 y = 0 \quad (6)$$

has solutions

$$y_1 = e^{i\omega t}, \quad y_2 = e^{-i\omega t}, \quad (7)$$

and they form a FS.

Q1: From Lec. 10 we know that (6) has a FS $\{\cos \omega t, \sin \omega t\}$. So, how are the complex sol's (7) related to the real sol's $\{\cos \omega t, \sin \omega t\}$?

Q2: In general, what does a complex solution $e^{(\alpha + i\beta)t}$ have to do with real solutions of a DE?

16 Definition of the complex exponential.

In Calculus II, you learned the Maclaurin (or Taylor) series for e^x :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (8)$$

(14-3)

It was proved in Calc. II that this series converges for all real x .

In a higher-level course (Complex Analysis) one proves that it actually converges for all complex x , too, i.e. when

$$x = a + ib.$$

Moreover, in Complex Analysis one also shows that for any x_1 & x_2 , one has the familiar rule:

$$e^{x_1+x_2} = e^{x_1} \cdot e^{x_2} \quad (9)$$

even when x_1, x_2 are complex.

Therefore,

$$e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} \quad (10)$$

Since we know how to compute $e^{\alpha t}$, we just need $e^{i\beta t}$.

1d Euler's formula

We first create a multiplication table for i :

n	0	1	2	3	4	5	6	
i^n	1	i	-1	$-i$	1	i	-1	 repeats!
			$\uparrow$	$\uparrow$		$\uparrow$	$\uparrow$	
			$(i)^2 = -1$	$(i)^3 \cdot i$	$i^4 = i^2 \cdot i^2$	$i^4 \cdot i$	$i^5 \cdot i$	

Now use (8):

$$e^{i\beta t} = 1 + \frac{i\beta t}{1!} + \frac{(i\beta t)^2}{2!} + \frac{(i\beta t)^3}{3!} + \frac{(i\beta t)^4}{4!} + \frac{(i\beta t)^5}{5!} + \frac{(i\beta t)^6}{6!} + \dots$$

use Table for i^n

$$\begin{aligned} &\cong 1 + i \cdot \frac{\beta t}{1!} - \frac{(\beta t)^2}{2!} - i \frac{(\beta t)^3}{3!} + \frac{(\beta t)^4}{4!} + i \frac{(\beta t)^5}{5!} - \frac{(\beta t)^6}{6!} + \dots \\ &= \underbrace{\left[1 - \frac{(\beta t)^2}{2!} + \frac{(\beta t)^4}{4!} - \frac{(\beta t)^6}{6!} + \dots \right]}_{\cos \beta t} + i \underbrace{\left[\beta t - \frac{(\beta t)^3}{3!} + \frac{(\beta t)^5}{5!} + \dots \right]}_{\sin \beta t} \end{aligned}$$

$$= \cos \beta t + i \cdot \sin \beta t.$$

So:

$$\boxed{e^{i\beta t} = \cos \beta t + i \sin \beta t} \quad (11)$$

Euler's formula

$$\begin{aligned} \text{Now: } e^{-i\beta t} &= e^{i(-\beta t)} = \cos(-\beta t) + i \sin(-\beta t) \\ &= \cos \beta t - i \sin \beta t \end{aligned}$$

$$\boxed{e^{-i\beta t} = \cos \beta t - i \sin \beta t} \quad (12)$$

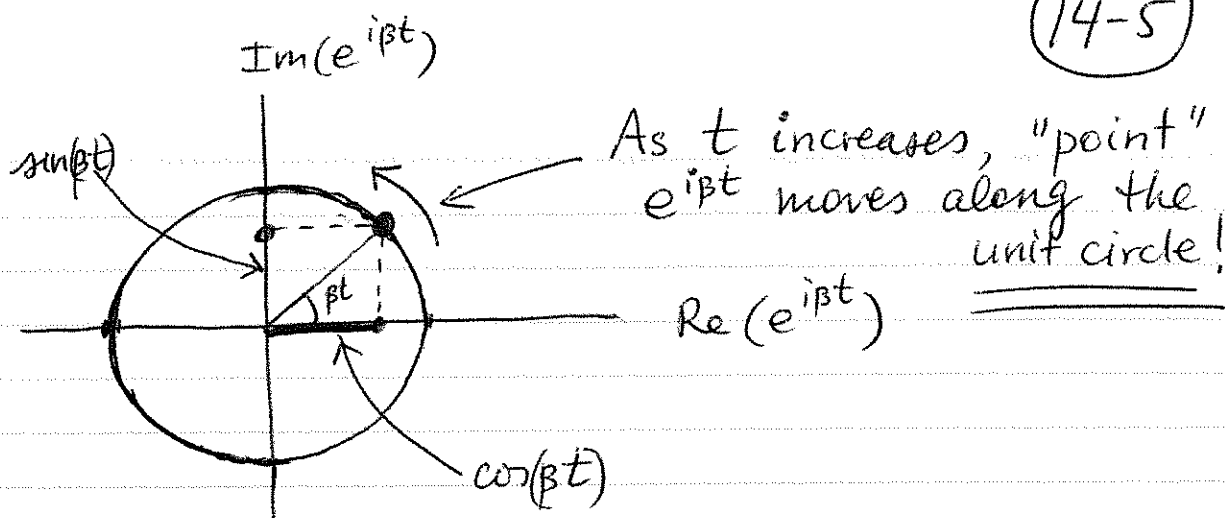
Ex. 2 Unit circle and Euler's formula

$$e^{i\beta t} = \underbrace{\cos \beta t}_{\text{real part}} + i \cdot \underbrace{\sin \beta t}_{\text{imaginary part}}$$

$\text{Re}(e^{i\beta t}) \qquad \text{Im}(e^{i\beta t})$

$$\text{Re}(a+ib) \equiv a; \quad \text{Im}(a+ib) \equiv b.$$

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(2) Constructing a real FS from complex exponentials

2a Method 1

$$y_1 = e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t} (\cos \beta t + i \sin \beta t) \quad (13a)$$

$$y_2 = e^{\alpha t - i\beta t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t} (\cos \beta t - i \sin \beta t) \quad (13b)$$

Since $\{y_1, y_2\}$ is a FS, then

$$z_1 = \frac{y_1 + y_2}{2}, \quad z_2 = \frac{y_1 - y_2}{2i} \quad \text{are also sol'ns}$$

of DE (3) (because any $c_1 y_1 + c_2 y_2$ is a solution).

Thus: $\boxed{z_1 = e^{\alpha t} \cos \beta t; \quad z_2 = e^{\alpha t} \sin \beta t} \quad (14)$

is another pair of solutions of (3).

Q! Do they form a FS?

A! Yes, See top of p. 136.

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26 Method 2

Thm. 3.3 (simplified version for DE-2 with constant coefficients; see book for general case).

Consider DE (3):

$$ay'' + by' + cy = 0 \quad (3)$$

with $a, b, c =$ real constants

Let

$$y = e^{xt+ipt} = e^{xt}(\cos pt + i \sin pt)$$

$$= \underbrace{e^{xt} \cos pt}_{u(t)} + i \underbrace{e^{xt} \sin pt}_{v(t)}$$

be a sol'n of (3). Then $u(t)$ and $v(t)$ individually are also sol'ns of (3).

Proof: $ay'' + by' + cy = 0$

$$a(\underline{u+iv})'' + b(\underline{u+iv})' + c \cdot (\underline{u+iv}) = 0$$

$$\underbrace{(au'' + bu' + cu)}_{\text{real}} + i \underbrace{(av'' + bv' + cv)}_{\text{purely imaginary}} = 0$$

This can hold only when

"real = 0" and "imaginary = 0"

$$\Rightarrow au'' + bu' + cu = 0 \quad \text{and}$$

$$av'' + bv' + cv = 0.$$

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Thus, we've shown that

$$u = e^{\alpha t} \cdot \cos \beta t, \quad v = e^{\alpha t} \cdot \sin \beta t$$

are sol'ns of (3).

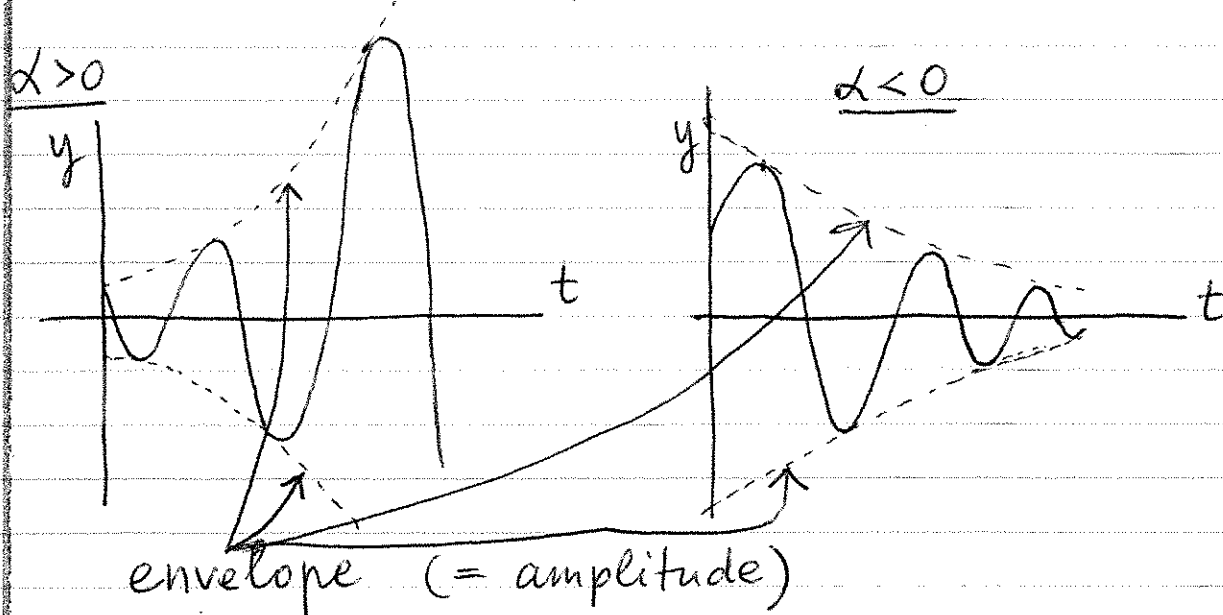
2c

The general solution

$$\begin{aligned} y &= c_1 \overset{u}{z_1} + c_2 \overset{v}{z_2} \\ &= c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t \\ &\equiv A e^{\alpha t} \cos \beta t + B e^{\alpha t} \sin \beta t. \end{aligned} \quad (15)$$

See Ex. 2 in book for using this general sol'n to solve an IVP.

Note: Let us plot a representative solution (for some A & B)



③ Amplitude and phase of solution

Trig. identity:

$$\cos(a-b) = \cos a \cdot \cos b (+) \sin a \cdot \sin b$$

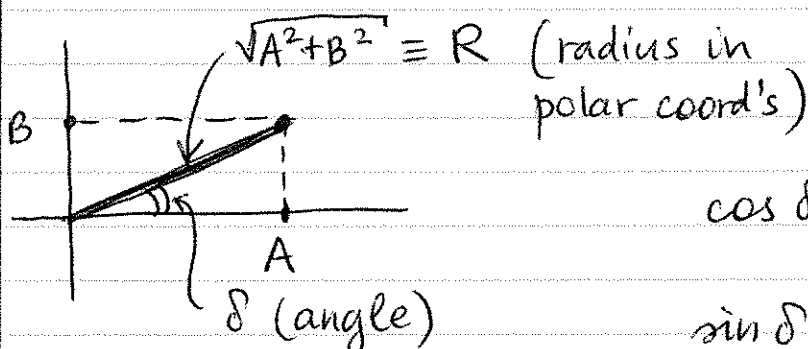
Sanity check: $a=b$

note the sign!

$$\underbrace{\cos(a-a)}_1 = \underbrace{\cos a \cdot \cos a}_{\cos^2 a} + \underbrace{\sin a \cdot \sin a}_{\sin^2 a} \quad \checkmark$$

Now let us "walk backwards" from r.h.s. to l.h.s.

$$A \cos \beta t + B \sin \beta t = \sqrt{A^2+B^2} \left(\frac{A}{\sqrt{A^2+B^2}} \cos \beta t + \frac{B}{\sqrt{A^2+B^2}} \sin \beta t \right)$$



$$\boxed{\tan \delta = \frac{B}{A}}$$

$$\cos \delta = \frac{A}{\sqrt{A^2+B^2}}$$

$$\sin \delta = \frac{B}{\sqrt{A^2+B^2}}$$

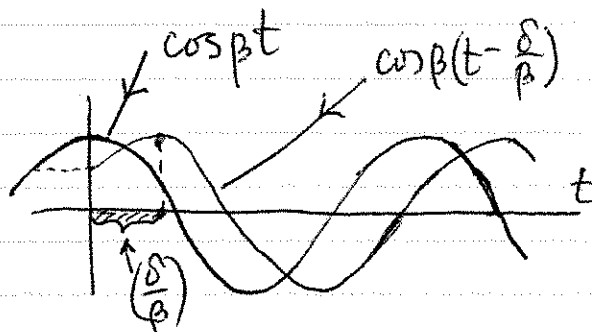
Then

$$\frac{A}{\sqrt{A^2+B^2}} \cos \beta t + \frac{B}{\sqrt{A^2+B^2}} \sin \beta t = \cos \delta \cos \beta t + \sin \delta \sin \beta t$$

$$= \cos(\beta t - \delta)$$

Meaning of "- δ "

$$\cos(\beta t - \delta) = \cos\left(\beta \left[t - \frac{\delta}{\beta}\right]\right)$$



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So $(\frac{\delta}{\beta})$ represents the "phase shift" of the cosine function.

Note: the book incorrectly refers to δ as the phase!

Now look at the general sol'n of DE (3):

$$y = e^{\alpha t} (A \cos \beta t + B \sin \beta t)$$

$$= e^{\alpha t} \cdot \sqrt{A^2 + B^2} \cdot \left(\frac{A}{\sqrt{A^2 + B^2}} \cos \beta t + \frac{B}{\sqrt{A^2 + B^2}} \sin \beta t \right)$$

$$= e^{\alpha t} \cdot R \cos(\beta t - \delta) = e^{\alpha t} \cdot R \cdot \cos \beta \left(t - \frac{\delta}{\beta} \right).$$

Ex. 3(a) Rewrite

$$y = \sqrt{3} \cdot \cos \frac{t}{2} + \sin \frac{t}{2}$$

in the form $R \cos(\beta t - \delta)$ and sketch it.

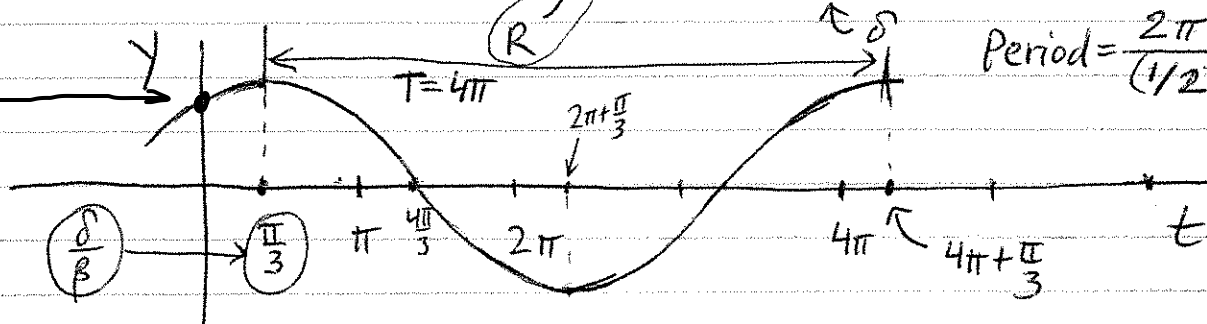
Sol'n:

$$\delta = \arctan \frac{B}{A} = \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

$$R = \sqrt{A^2 + B^2} = \sqrt{3 + 1} = 2.$$

$$y = 2 \cos \left(\frac{t}{2} - \frac{\pi}{6} \right) = 2 \cos \left[\frac{1}{2} \left(t - \frac{\pi}{3} \right) \right]$$

$$\text{Period} = \frac{2\pi}{(1/2)}$$



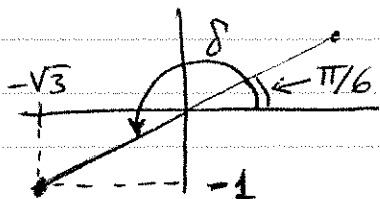
Check: Must agree with $y(0) = \sqrt{3} \cdot \cos 0 + \sin 0 = \sqrt{3}.$

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Ex. 3(b) Same for

$$y = -\sqrt{3} \cos \frac{t}{2} - \sin \frac{t}{2}.$$

Sol'n:



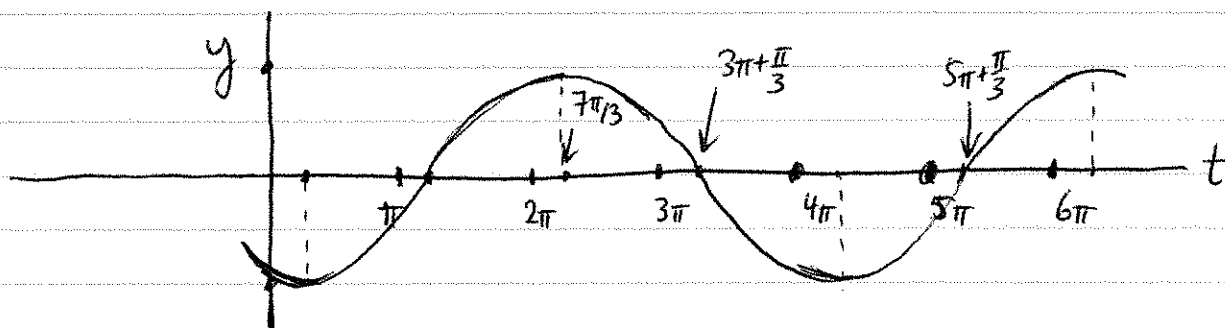
$$\delta = \arctan\left(\frac{-1}{-\sqrt{3}}\right) + \pi$$

$$= \frac{\pi}{6} + \pi,$$

↑
need to
add when
 $A < 0$

$$y = 2 \cos\left(\frac{1}{2}t - \left(\frac{\pi}{6} + \pi\right)\right) = 2 \cos\left[\frac{1}{2}\left(t - \frac{7\pi}{3}\right)\right]$$

as in 3(a)



Not surprisingly, this looks like the graph in Ex. 3(a), but flipped about the x -axis.

HW

Sec. 3.5.

1, 2 - polar to Cartesian; use Complex Expand

3, 7, 9, 11 ← find $\lambda = \alpha + i\beta$, gen. sol, sol. of IVP

13, 15, 19, 21 ← from given sol'n find a, b, y_0, y_0' . (19, 21 - expand) (the cos first)

23, 24, 25, 26 ← rewrite $A \cos t + B \sin t$ as $R \cos(t - \delta)$.

(for 24, 26 - use Trig Factor)

27, 28, 29 - rewrite as $R \cos(\beta t - \delta)$ from graph (#28: $\cos(\frac{3}{2}t - \pi/8)$) amplitude

Hint: 1) find period T ; 2) $\beta = \frac{2\pi}{T}$ (Lec. 10); 3) find δ/β ; 4) $R \cos(\beta t - \delta)$.

31 ← find λ_1, λ_2 and use f.l.a for W from top of p. 124.

33 ← roots of complex DE; Rey is not a sol'n. || 32 ← buoyancy w/drag.