where $\mathbf{v}$ and $\mathbf{b}$ are constant vectors. If $\mathbf{v} \neq \mathbf{0}$, then the particle travels in a line with constant velocity $\dot{\mathbf{r}}(t) = \mathbf{v}$. On the other hand, if $\mathbf{v} = \mathbf{0}$, then the particle is motionless at position $\mathbf{b}$.

We will model planetary motion by assuming that the planet is the particle and that it is acted upon only by the sun, which we place at the origin. Newton conceived of the sun as an attractor that draws the planet toward itself, causing it to orbit the sun in a characteristic curve. He supposed that the force drawing the planet toward the sun depended on the distance r between the planet and the sun. This idea of a *central* force may be expressed in vector notation in the following way:

$$\ddot{\mathbf{r}} = -f(r)\frac{\mathbf{r}}{r};$$

that is, a central force is directed toward the origin and depends only on the distance from the origin. The simple fact that the force is central has two extraordinary consequences. First, if the force is central, then by simple properties of the cross product we obtain

$$\mathbf{r} \times \ddot{\mathbf{r}} = -\frac{f(r)}{r}\mathbf{r} \times \mathbf{r} = \mathbf{0},$$

and hence,

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

The vector $\mathbf{r} \times \dot{\mathbf{r}}$ is called the *angular momentum* of the particle. This equation says that under the action of a central force angular momentum is conserved, i.e.,

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c},$$

for some constant vector $\mathbf{c}$. What is the geometrical meaning of this? The vector $\dot{\mathbf{r}}$ is tangent to the path of motion, so the equation above says that the path of motion lies in the plane through the origin, which is perpendicular to $\mathbf{c}$ (an algebraic proof uses the triple product identity: $\mathbf{r}\cdot\mathbf{c} = \mathbf{r}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = (\mathbf{r}\times\mathbf{r})\cdot\dot{\mathbf{r}} = 0$). So, under the influence of a central force, motion is *planar*. This simplifies both the visualization and the analysis of centrally directed motion.

It is convenient to picture planar motion in a polar coordinate system, that is,

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$$

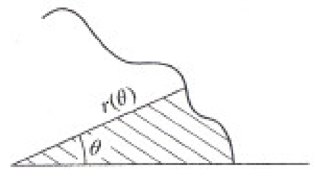

Figure 3.4: Areas in Polar Coordinates

(see Figure 3.4). Consider now the formula for the area A swept out by the radius vector as the angle varies from 0 to θ:

$$A = \frac{1}{2}\int_0^\theta r(\alpha)^2 d\alpha.$$

The *rate* at which area is swept out is then, by the Fundamental Theorem of Calculus,

$$\frac{dA}{dt} = \frac{1}{2}r(\theta)^2\dot{\theta}.$$

However,

$$\dot{\mathbf{r}} = (\dot{r}\cos\theta - r\sin\theta\dot{\theta})\mathbf{i} + (\dot{r}\sin\theta + r\cos\theta\dot{\theta})\mathbf{j},$$

and hence, by conservation of angular momentum,

$$\mathbf{c} = \mathbf{r} \times \dot{\mathbf{r}} = r^2\dot{\theta}\mathbf{k} = \frac{2}{c}\frac{dA}{dt}\mathbf{c},$$

that is, dA/dt is a constant, namely $c/2$.

In other words, under the influence of a central force, the radius vector sweeps out equal areas in equal times. This is Kepler's second law.

Recall that a conic section is characterized as a plane curve, which is the locus of a point the ratio of whose distance from a fixed point O (a *focus*) and from a fixed line L (the *directrix*) is a constant e (the *eccentricity*). An analytical representation of the general conic section in polar coordinates is easy to come by. We put the origin at O and take the polar axis perpendicular to L, as in Figure 3.5. Denote the distance from L to O by k.

The condition $|OP| = e|PQ|$ then becomes

$$r = ek - er\cos\theta,$$

or

$$r = \frac{ek}{1 + e\cos\theta}.$$

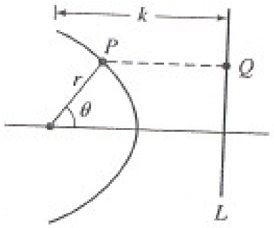

Figure 3.5: A Conic Section

Suppose now that the body orbits under the influence of an inverse-square central force, that is, the position vector satisfies

$$\ddot{\mathbf{r}} = -\frac{a}{r^3}\mathbf{r},$$

where a is a constant. Since the force is central, we have seen that

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. By Problem 3, we find that

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{\mathbf{c} \times \mathbf{r}}{r^3}.$$

Multiplying by $-a$, we then have

$$-a\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \mathbf{c} \times \frac{-a\mathbf{r}}{r^3} = -\ddot{\mathbf{r}} \times \mathbf{c}.$$

Integrating this gives

$$a\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right) = \dot{\mathbf{r}} \times \mathbf{c},$$

where $\mathbf{e}$ is a constant of integration. Note that $\mathbf{e} \cdot \mathbf{c} = 0$, and hence $\mathbf{e}$ lies in the orbital plane. If we measure the angular displacement of the orbiting body with respect to the axis in direction $\mathbf{e}$, as in Figure 3.6, then it is easy to see that the orbit is a conic section. Indeed, from the last equation we obtain

$$a(\mathbf{r} \cdot \mathbf{e} + r) = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{c}) = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{c} = c^2.$$

But $\mathbf{r} \cdot \mathbf{e} = re \cos \phi$, so

$$r(1 + e \cos \phi) = c^2/a$$

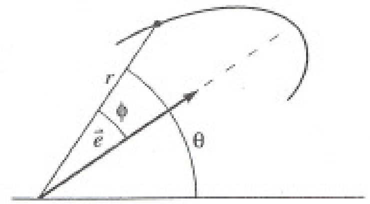

Figure 3.6: Orientation of the Orbit

or,

$$r = \frac{ek}{1 + e \cos \phi},$$

where $k = c^2/(ae)$, which is a conic section of eccentricity e with focus at the center of attraction. Depending on the value of e, the conic may be an ellipse, a parabola, or a hyperbola (see Problem 4). This is Kepler's first law.

It is useful at this point to summarize the main results for a body orbiting a center of attraction:

no force $\Longrightarrow$ uniform straight line motion (or rest)

central force $\Longrightarrow$ planar orbit + equal areas law

inverse square central force $\Longrightarrow$ conic orbit with focus at origin.

What Newton did was provide a firm mathematical justification for these observations of his predecessors (to be sure, his development was strictly Euclidean and looked nothing like the treatment given above). It is natural to ask about the corresponding inverse problems: To what extent may the implications "$\Longrightarrow$" be replaced by "$\Longleftarrow$?" These and other inverse problems were addressed by Newton in the *Principia*. The reader, equipped with the powerful techniques of vector calculus, is invited to investigate a number of such inverse problems in the following activities.

3.3.2 Activities

1. Exercise Show that in a uniform circular orbit

$$\mathbf{r} = r \cos at\mathbf{i} + r \sin at\mathbf{j},$$

the force is central and has magnitude $\|\dot{\mathbf{r}}\|^2/r$.

2. Exercise Verify the vector identity: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.

3. Problem Show that

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{(\mathbf{r} \times \dot{\mathbf{r}}) \times \mathbf{r}}{r^3}.$$

4. Problem Recall that an inverse-square central force leads to an orbit of the form

$$r = \frac{ek}{1 + e\cos\phi}, \quad ek = c/a^2.$$

Use more familiar rectangular coordinates to show that if $0 < e < 1$, the orbit is an ellipse, while if $e = 1$, the orbit is a parabola, and if $e > 1$, the orbit is a hyperbola.

5. Question Suppose a particle moves entirely in a plane containing the origin. Is the motivating force necessarily *central*?

6. Problem Give a geometrical proof that for uniform straight-line motion, equal areas are swept out by the radius vector in equal times.

7. Exercise Suppose a body moves in a straight line with constant speed. Show that no force acts on the body.

8. Exercise Use the equal areas law to show that under the influence of a central force,

$$2\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0.$$

9. Problem Show that a central force implies

$$r\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{r}.$$

10. Problem Show that if a body orbits in a plane containing the origin, and the radius vector sweeps out equal areas in equal times, then the body is motivated by a centrally directed force. (This is Proposition II, Theorem II of Book I of the *Principia*.)

For the remainder of this module, we assume a central force and use the notation $u = r^{-1}$. We assume that the body does not collide with the center of force and hence u is defined for all times.

11. Problem Use the equal areas law to show that $r\dot{\theta}^2 = h^2u^3$, where h is twice the rate at which area is swept out.

12. Problem Show that $\ddot{r} = -h^2u^2\left(\frac{d^2u}{d\theta^2}\right)$.

13. Exercise Conclude from the previous three problems that, under the influence of a central force,

$$\ddot{\mathbf{r}} = -g(u)\frac{\mathbf{r}}{r},$$

where

$$g(u) = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right).$$

This result provides a general tool for the inverse problem for orbits, as the magnitude of the central force is given by $f(r) = g(r^{-1})$. This can be used to find the force functions for a number of simple orbits.

14. Problem Find the central force on a body that orbits on a circular arc that passes through the center of force (*Principia*: Book I, Proposition VII, Corollary I).

15. Problem Show that if a body orbits in a conic with the center of force at a focus, then the force on the body is proportional to the reciprocal of the square of the distance between the body and the center of force (*Principia*, Book I, Propositions XI, XII, XIII).

16. Problem Find the central force on a body whose orbit is a logarithmic spiral, i.e., $r = e^{a\theta}$ (*Principia*, Book I, Proposition IX).

17. Problem Find the central force on a body that orbits in an Archimedean spiral, i.e., $r = a\theta$.

18. Problem Find the central force on a body that orbits in an ellipse whose *center* is the center of force (*Principia*, Book I, Proposition X).

19. Problem Find the central force on a body that orbits on the hyperbola $x^2 - y^2 = a^2$.

20. Problem Find the central force on an object that orbits on the lemniscate $r = a\sqrt{\cos 2\theta}, \quad 0 \le \theta < \pi/4$.

21. Computation Use the program 'ode23' in MATLAB, along with the m-file 'orbit0' provided, to investigate orbits under the constant force law $\ddot{\mathbf{r}} = -\mathbf{r}/r$. Plot the orbits for various initial values. The program is invoked as follows:

$$[t, z] = \text{ode23}('\text{orbit0}', t0, tf, z0),$$