MATH 235 — Mathematical Models in Science and Engineering

Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 2

1. This problem is worth **0.5 point**.
   Use the data in the table below and the approach based on logarithms, explained in the Lecture, to confirm Kepler's Third Law. Attach a graph supporting your work.
   Note: Substitute the mean distances provided in the table for the major semi-axes of the corresponding ellipses. The eccentricities are listed for your information only; they are not needed for your solution.

Planet	Period (Earth days)	Mean distance from Sun (km $\times 10^6$)	Eccentricity of orbit
Mercury	88	57.9	0.206
Venus	225	108.2	0.007
Earth	365	149.6	0.017
Mars	687	227.9	0.093
Jupiter	4329	778.3	0.048
Saturn	10,753	1427	0.056
Uranus	30,660	2870	0.046
Neptune	60,150	4497	0.010
Pluto	90,670	5907	0.249

2. Show that Eq. (2.11) reduces to the Cartesian equations in the standard form for:

   - an ellipse, when $0 < e < 1$ (*see also the Note (not the Hint!) below*);
   - a parabola, when $e = 1$; and
   - a hyperbola, when $e > 1$.

   Hint: Rewrite (2.11) as $r = eD - er\cos\theta$, square both sides, and use the relation between the polar and Cartesian coordinates. Then, complete the square and thereby put your equations in the form of the canonical equations of the ellipse and hyperbola:

$$\frac{(x - x_0)^2}{a^2} \pm \frac{(y - y_0)^2}{b^2} = 1, \qquad \text{(HW2.1)}$$

   where '$+$' and '$-$' pertain to the ellipse and hyperbola, respectively. In the case of an ellipse, a and b are its semi-axes.

For the ellipse only, you also need to do the following:
(i) Find its geometric center, i.e. the values (x_0, y_0) in the equation above.
(ii) Find its major and minor axes (i.e., a and b) in terms of e and D.
(iii) The case of $e = 0$ yields a circle, because you will find that in this case $a = b$. However, if you simply set $e = 0$ in your equation, you will obtain a circle of radius 0. To obtain a circle of a nonzero radius, you should let $e \to 0$ but $eD \to C$, where C is some nonzero constant. How is this C related to the radius of the circle? Where is the center of this circle?
You will use these results in the next problem.

Note: As we have already mentioned, in the ellipse described by Eq. (HW2.1), the geometric center is at (x_0, y_0). Then the origin in that equation is at one of the foci of the ellipse. In the next problem, we will refer to this curve as the *original ellipse*. Conversely, if one shifts this ellipse so that now its geometric center would coincide with the origin, then the foci of this shifted ellipse will be at (x_0, y_0) and $(-x_0, -y_0)$. (Make a sketch to convince yourself that this is true.) In the next problem, we will refer to this curve as the *shifted ellipse*.
Thus, to summarize: The *original ellipse* has is focus at the origin, while the *shifted ellipse* has its geometric center at the origin.

3. This problem is worth **0.5 point**.
As you may recall from the Lecture, Kepler was struggling with the fact that Tycho Brahe's observations showed that Mars' orbit is not a circle. (The fact that the Sun is not located at the center of the orbit had already been known to Ptolemy 1500 years before Kepler.) A simple exercise below is intended to make you appreciate how small was the effect on which Kepler has spent so much effort trying to find a model for it. You may find details about Kepler's discovery in a link posted alongside Lecture 2.

   Use any computer program to make the following plots *in the same figure* using Eq. (2.11).
   (i) A plot of the *shifted ellipse* (see the *Note* at the end of Problem 2) with $e = e_{\rm Mars}$, an arbitrary D.
   (ii) A plot of a circle with the radius equal to the major semi-axis of the ellipse in (i) and also centered at the origin.
   Mark (by hand is okay) which curve is the ellipse and which is the circle.
   In addition, you should plot the center of the circle from (ii) (i.e., the point $(0, 0)$) and a focus of the *shifted ellipse* from (i) (i.e., the point (x_0, y_0); see the *Note* at the end of Problem 2). See the *Technical note* below on how one can plot ellipses and points.

   Last question: In percent (relative to the major axis of the ellipse), how small is the difference between the major and minor axes that Kepler was concerned with?

   Technical note: Ellipses (and circles) are easiest to plot using their parametric equations, which for the canonical equation (HW2.1) are: $x = x_0 + a\cos t$, $y = y_0 + b\sin t$. In Matlab, you need to first define a vector `t=[0:0.01:2*pi]` and then plot `y` versus `x` using the regular command `plot`. Type `help plot` for details. In Mathematica, you need to use the command `ParametricPlot`; see a Help entry for it if needed. To plot a point in Matlab, mimic this example:
   ```
   plot(1,2,'ro','markersize',6)
   ```
   To plot a point in Mathematica, use the command `Point`; see a Help entry for it if needed.

4. Derive Kepler's Third Law following the steps below.
   1) Use the formula $A = \frac{1}{2}\int_{\theta_{\rm min}}^{\theta_{\rm max}} r^2 d\theta$ (see the first equation in Sec. 2.2.1) and Eq. (2.20) to find the area of a planet's elliptical orbit. (What are the $\theta_{\rm min}$ and $\theta_{\rm max}$ is this case?) Use *Mathematica* to do the integration.

Note about notations: In (2.20), N is the proportionality coefficient in Newton's Law of Gravitation (2.12). Thus, (2.20) supplies a relation between parameters N and $c \equiv |\vec{c}|$ (defined in (2.14) and earlier in (2.1)) one one hand and the ellipse's parameters e and D (see (2.11)).

Technical note: You need to tell *Mathematica* that $0 < e < 1$ (otherwise the orbit is not an ellipse and hence the integral could not be found). The way to do it is by using the option `Assumptions`, as shown below:

`Integrate[f[theta],{theta,0,2*Pi},Assumptions -> {e>0,e<1}],`

where `f[theta]` is the function you want to integrate.

2) Note that according to Eq. (2.1), $\dot{A} = c/2$. Since by Kepler's Second Law, the planet's radius-vector sweeps out equal areas in equal intervals of time, the period of the planet is found simply as $T = A/\dot{A}$. Find this T.

3) In Problem 2 you should have expressed the major axis, $2a$, of an ellipse via the parameters D and e. State your answer here. (If you could not do this part of Problem 2, ask me or Google for an answer.)

4) Combine the results of steps 1) – 3) to obtain Eq. (2.21) of the notes. *Make sure to demonstrate* that the constant on the right-hand side (r.h.s.) does not depend on the eccentricity of the orbit and on the revolution rate c. (This is what makes it a "constant", i.e., the same for all planets, in the Third law.)

5. In the Lecture, we showed that given the Universal Law of Gravitation, Eq. (2.12), one can find the equation of motion $r = r(\theta)$ (Eq. (2.20)).[1] Here we address the inverse problem: Given the equation $r = r(\theta)$ of the orbit, how can we find the *central* force $\vec{\mathbf{F}} = f(r)\,\vec{\mathbf{r}}/r$? This inverse problem was also solved by Newton in his treatise "Principia Mathematica".

   Here you will be guided towards the solution of the inverse problem following the ideas of Problems 8, 9, 11 – 13 on p. 74 of the book "Inverse Problems" by C.W. Groetsch; these problems are posted alongside this HW assignment.

   1) Take the cross product of Eq. (2.15) with $\vec{r}$ and then use Eq. (2.7) to show that

$$r^2\ddot{\vec{\mathbf{r}}} = \left(\ddot{\vec{\mathbf{r}}} \cdot \vec{\mathbf{r}}\right)\vec{\mathbf{r}}\,.$$

   Apply the Product Rule to $(\dot{\vec{\mathbf{r}}} \cdot \vec{\mathbf{r}})^{\cdot}$ and "massage" the result to show that the above equation becomes:

$$r^2\ddot{\vec{\mathbf{r}}} = \left(\left(\dot{\vec{\mathbf{r}}} \cdot \vec{\mathbf{r}}\right)^{\cdot} - \left(\dot{\vec{\mathbf{r}}} \cdot \dot{\vec{\mathbf{r}}}\right)\right)\vec{\mathbf{r}}\,.$$

   Obtain the expression for the first term on the r.h.s. by taking the derivative of the third equation in (2.9). For the second term, recognize that $\dot{\vec{r}} = \langle \dot{x}, \dot{y} \rangle$ and then use Eqs. (2.3). Combine these results to obtain the formula of Problem 9 on p. 74 of Groetsch's book.

   Note that this yields the equation of motion in the form

$$\ddot{\vec{\mathbf{r}}} = f(r, \ddot{r}, \dot{\theta})\,\frac{\vec{\mathbf{r}}}{r}\,.$$

   Our next goal is to eliminate the time derivatives from $f(r, \ddot{r}, \dot{\theta})$ and express it as a function of the sole variable r. You will do this in the next two steps.

   First, in Step 2), you will express $\dot{\theta}$ as a function of r and then express $\ddot{r}$ as a function of θ. Now, since r is given as a function of θ, one can, at least in principle, express θ as a function of r and substitute the result into the expression for $f(\,r,\, \ddot{r}(\theta),\, \dot{\theta}(r)\,)$ to obtain f as a function of r only.

[1] In fact, it is possible to find $r = r(\theta)$ for a force of the form $\vec{\mathbf{F}} = \vec{\mathbf{r}}/r^n$ with any n, not just $n = 3$.

2) Let us recall from Eq. (2.1) that $r^2\dot\theta = c = \text{const}$. Hence $\dot\theta = c/r^2$.

To continue with finding $\ddot r$, note that on an orbit traced by a planet, $r = r(\theta)$, where $\theta = \theta(t)$. That is, $r = r\Big(\theta(t)\Big)$. Then use the Chain Rule to show that

$$\dot r = \frac{dr}{d\theta}\frac{c}{r^2}\,.$$

Define $u(\theta) \equiv 1/r(\theta)$ and rewrite *only* the r.h.s. of the above equation in terms of u and/or $du/d\theta$.

To finally obtain an expression for $\ddot r = (\dot r)^{\cdot}$, use the Chain Rule again. Now obtain the result of Problem 12 on p. 74 of the book. How is h related to c?

3) Combine the results of 1) and 2) to obtain the formula of Problem 13 on p. 75. Note that now you have found $f(r, \ddot r, \dot\theta)$ in the form $f(u(\theta), d^2u(\theta)/d\theta^2)$, where $u = 1/r$. Examples of using this result to obtain specific expressions for f in terms of r only, are considered in the next problem.

6. This problem is worth **1.5 points**.
   Use the result of Problem 5 above to do the following Problems from p. 75 of the book by C.W. Groetsch: (a) # 15 (here the "conic" is the curve described by Eq. (2.11) of the Notes), (b) # 16, (c) # 14 (yes, I suggest that you do the problems in this order). See the *Notes* below.
   General Note: You can do this Problem independently of Problem 5 above; all you need to use here is the result of that Problem.
   Note for (a): You may recognize # 15 as being the inverse problem of the derivation of Kepler's First law in Sec. 2.4 of the lecture notes.
   Note for (c): The posted Hint (from Groetsch's book) gives you the equation for $u(\theta)$ and then guides you through subsequent algebra; so all that remains for you to do is to fill in minor details. To make this problem a little more substantial, you also *must do* the following:
   Plot or sketch the circle corresponding to the formula in the Hint and state how the constant a in that formula is related to the circle's radius.

7. Create a folder named something like `math235` in the drive where you have Matlab installed (your share of the M-drive if you use one of the UVM computers and C-drive if you have Matlab installed on your personal computer). Download files `hw2_p7.m` and `hw2_p7_funorbit.m` from the homework website into this folder.

   Vary *only* two parameters: `tf` and `vy0`, in `hw2_p7.m` and run it to obtain an elliptical and a hyperbolic orbits. Also, try to get as close to a parabolic orbit as you can.

   Hint: Before doing this problem, take a piece of paper and make a sketch showing the location of the Sun and the initial location of the planet, as given in lines 11 and 12 of the code `hw2_p7.m`. Now think in what direction the planet should move at the initial moment to make a closed orbit (i.e., an ellipse) possible. What about a non-closed orbit (hyperbola)?

   Attach plots of curves that you believe to be an elliptical, hyperbolic, and parabolic orbits. On each plot, write the values of `tf` and `vy0` for which it was obtained.

   Since a parabolic orbit looks very much like a hyperbolic one, write a brief explanation as to why you believe that your graph is a piece of a parabola rather than that of a hyperbola. (Base your explanation on the difference in the *asymptotes* of parabolas and hyperbolas. You must have learned about these asymptotes in Calculus III; if not, or if you do not remember it, find the information in a textbook or online.)

Bonus 1 This problem is worth **0.5 point**.

Change the initial condition in `hw2_p7.m` to be $(0,\ 2)$ and repeat the assignment of Problem 7.

Note: The main challenge here is to find a parabolic orbit and *convincingly* demonstrate that it is neither an arc of an ellipse nor a hyperbola.

Hint: You may need to vary both components of the initial velocity.

Bonus 2 This problem is worth **0.5 point**.

If you have been able to obtain an elliptical orbit in Bonus 1, you may notice that its major axis is not horizontal (unlike in Problem 7). Worst yet, the origin, which is the center of the modeled gravitational force, is clearly not at the ellipse's focus. At this point you may begin thinking one or more of the following:

(i) I've just disproved Kepler's First Law and its derivation from Newton's Laws. This sounds like the Nobel Prize in Physics.

(ii) I'm losing my mind.

(iii) The instructor has given us an incorrect code to simulate the orbit.

(iv) Something else is going on.

Upon some reflection, you may conclude that the answers to the first two possibilities may be 'not yet' (although one may still be more likely than the other). As far as the third possibility, this instructor has spent about an hour checking his code in various ways after he had noticed the fact stated in the second sentence of this problem. The code is correct. Thus, it must be the fourth possibility.

The reason for the observed discrepancy is painfully simple. Examine the `axis` property of the `plot` command in Matlab: type `help axis`. Draw a conclusion from what you have read (you will need only one piece of information from there), and amend the plotting command so that the resulting plot would agree with Kepler's First Law.

Explain how you have amended the plotting command; attach the modified plot; and convince me that it no longer violates the Kepler Law.

Bonus 3 In Problem 7 you were asked to provide a qualitative explanation whether the observed trajectory is a parabola or a hyperbola. Here you are asked to *rigorously* predict for what value of `v0y` in `hw2_p7.m` the trajectory will be a parabola. You will need to do so by computing the value of eccentricity e from the other parameters given in this problem. Follow these steps.

1) Look inside the code `hw2_p7.m` and determine the initial values: $r(0)$, $\theta(0)$, $\dot{x}(0)$, $\dot{y}(0)$.

2) Using the relation between Cartesian and polar coordinates, express $\dot{x}$ and $\dot{y}$ in terms of $\dot{r}$ and $\dot{\theta}$ at all times. Now, substitute in these two equations the initial values found in Step 1).

3) Use Eq. (2.11) of the Notes to relate $\dot{r}$ to $\dot{\theta}$; now, using the initial values, relate $\dot{r}(0)$ to $\dot{\theta}(0)$. In the process, you will need a value for eD; this is found from the initial conditions for r and θ.

4) Finally, substitute $\dot{r}(0)$, expressed via $\dot{\theta}(0)$, into the equations for $\dot{x}(0)$, $\dot{y}(0)$ obtained in Step 2). For what value of `v0y` will you obtain the value of e corresponding to a parabola?