MATH 235 — Mathematical Models in Science and Engineering

Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 3

When doing the two problems below, follow the more detailed guidelines found in Section 3.3. The numeration of the equations refers to Lecture 3 (that is, Eq. (23a) quoted below stands for Eq. (3.23) in the notes, etc.).

Note the following difference between the equations of motion of the linear and quadratic models of air resistance. The linear model is governed by one equation, (5). Therefore, the solutions v and y for that model, obtained in the notes, applied to both upward and downward motions. In contrast, the quadratic model requires two different equations, (22a) and (22b), for going up and going down. This is reflected below in that Problems 1 and 2 treat separately the upward and downward motions, respectively.

1. This problem is worth **4 points**. Weight of each part is shown next to it. Parts (b)–(f) refer to the "first approach" (pp. 24–26 of Section 3.2), while parts (g)–(i) refer to the "second approach" (p. 26).

   (a) (**0.75**) Obtain (23a).
   Note: Here and in the remainder of this part, you must do all the required integrations by hand (i.e., *not* by Mathematica). Also, you are allowed to use only the most basic table integrals, such as, e.g., $\int du/u = \ln u$ (the usual "$+C$" is implied) or $\int du/(1+u^2) = \arctan u$. However, you are not allowed to use table expression for, e.g., $\int du/(a^2+u^2)$ with $a^2 \neq 1$ or for $\int \tan u\, du$. You must derive those expressions using a u-substitution or, perhaps, several of those.

   Next, solve (23a) for v and integrate the resulting expression to obtain (23b).
   Finally, obtain (24) from (23a).

   Suggestion: When integrating (23a) over τ, denote the constant $\arctan(\sqrt{k}v_0)$ by, say, X, and carry out the calculation using that notation. Substitute back $X = \arctan(\sqrt{k}v_0)$ only at the last step of the calculation. The same suggestion applies to other calculations when appropriate.

   (b) (**0.5**) Assuming $y_0 = 0$ in (23b), find $y_m \equiv y(\tau_m)$. Now, find the first two terms of the Taylor (Maclaurin) expansion of y_m in powers of K. You will need this result in Problem 2.
   Hint: To find the Maclaurin series for $\ln\sqrt{1+z}$, first use a well-known property of logarithms and then use the Maclaurin expansion
   $$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots .$$

   (c) (**0.25**) You will be asked to derive (25a) later. For now, assume that it is valid and integrate (25a) to obtain (25b).

   (d) (**0.5**) Following the approach illustrated by Eqs. (13) and (14), determine when expansion (25a) is valid. Your solution must address all the points addressed in the Lecture notes and must show all necessary details. (In particular, you must examine restrictions on both the initial velocity v_0 and the times τ and determine whether they lead to the same condition for the drag coefficient K.)

(e) (**0.25**) Use Mathematica to Taylor-expand (23b) to confirm that the result coincides with (25b).
Note: To find out which command to use, look in the Mathematica's Help browser for "Taylor" (without the quotes). Mathematica will guide you to the correct command. A good idea is to read not only the general description (and often, not so much it), but the rubric Examples, found after the general description.

(f) (**0.25**) Obtain (26) by directly Taylor-expanding (24).

(g) (**0.25**) Taylor-expand (23a) to obtain (25a), as explained in the text between (24) and the sentence following (25a).

(h) (**0.5**) Re-obtain (25a) from (22a) by following the steps of the derivation of Eqs. (16).

(i) (**0.5**) Re-obtain (26) from (25a) by using (18) and following the steps of the derivation of Eq. (19).

2. This problem is worth **3.25 points**. Weight of each part is shown next to it. Parts (c), (d) refer to the "first approach" of Section 3.2, while parts (e), (f) refer to the "second approach".

(a) (**0.75**) Doing all the calculations by hand[1], derive (27a,b) and then (28a); also, find the value of C.
Hint 1: To derive (27a), use partial fractions.
Hint 2: To derive (27b) from (27a), use the following observation. To find X from $(1+X)/(1-X)=f$, multiply both sides of this equation by $(1-X)$ and then solve for X.

(b) (**0.5**) Integrating (28a) by hand, obtain (28b). Using the value of y_m which you obtained in Problem 1(b), verify that your (28b) is indeed equivalent to (29). Finally, obtain (30).

(c) (**0.5**) Obtain (31a,b) as explained in the text preceding these equations (i.e., starting with (27a)). When deriving (31b), you must explain why the integration constant is y_m.
Hint: To begin, use an identity for $\ln(a/b)$ (where you need to decide what a and b are); then expand each of the terms in powers of $\sqrt{K}v$. Only after that should you substitute (15).

(d) (**0.5**) Obtain (31c) from (30) by using Taylor expansion.
Note: When expanding $\ln(\sqrt{1+x^2}+x)$ for $x \ll 1$, proceed in three steps.
1) In a textbook or online, find the Binomial series for $\sqrt{1+z}$:

$$\sqrt{1+z} = 1 + a_1 z + a_2 z^2 + \ldots$$

2) Use the result of the previous step to find the Maclaurin series for the argument of the logarithm:

$$\sqrt{1+x^2} + x = 1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots \equiv 1 + y\,,$$

where you need to relate b_1, b_2, etc. to a_1, a_2, etc.. In the next step, you will have to decide how many terms in this expansion you will need to keep (this may have to be done by trial and error).
3) Expand $\ln(1+y)$ in a Maclaurin series, which can be found in Lecture 3. Note that for y, you need to substitute *not* just $b_1 x$, but the entire expression $(b_1 x + b_2 x^2 + b_3 x^3 + \ldots)$. Of course, you do not need to square (or cube, etc) this infinite series, but need to keep only as many terms as required to perform the calculations with the accuracy specified on the r.h.s. of (31c). Figuring out how many terms you need to keep is important for you to obtain the correct answer.

(e) (**0.5**) Re-obtain (31a) by substituting (15) into (22b). Justify your integration constants at each step.

(f) (**0.5**) Re-obtain (31c) from (31b), as explained in the final paragraph of the Lecture. You will need the Taylor expansion for y_m which you have obtained in Problem 1(b).

[1] See the *Note* for Problem 1(a).