MATH 235 — Mathematical Models in Science and Engineering

Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 4

1. This exercise intends to illustrate the following point: If initially you do not know how a given physical system will behave, you may develop some intuition for it by running numerical simulations. In this respect a numerical experiment is no different from a physical one.

   Download the codes `hw4_p2.m`[1] and `hw4_p2_track.m` from the Homework webpage.
   As explained below, you will need to run the code `hw4_p2.m` to observe the motion of the car on the track. The other code is auxiliary: when `hw4_p2.m` runs, it automatically uses `hw4_p2_track.m`, so you should not worry about it.

   When the car stays on the track, it is depicted as green, and when it flies off, it turns red. Watch the title of the figure; it is different depending on whether the car is on the track or not. Note that once the car has left the track, the code assumes that it will move under gravity until it hits the ground. That is, it can "go through" the track in its free-fall motion.

   You will need to change only three parameters in `hw4_p2.m` to do this problem: `b` (line 14), `track_type` (line 16), and `V0` (line 23). Read about their meanings in the comments inside the code.

   (a) Set `track_type=1` and `b=0.1` and then run `hw4_p2.m`. Your task is to find the *minimum* and *maximum* initial speeds `V0` for which the car will go over the top but will be on the track at all times. Find the answers with accuracy 0.1 (e.g., $V0 = 7.8$).

   Suggestion: Look at the title of the figure before the car begins its motion. This may save you time in that you may not need to run some of the simulations since you should be able to predict their outcome right from the start. Do not worry if you have not understood what this means at this point. Keep doing the simulations; this suggestion will make sense as you proceed.

   Technical note: If you need to abort the execution of the code before it ends, press `CTRL-C` (sometimes repeatedly).

   (b) Repeat part (a) for `track_type=2`. Compare the answers for parts (a) and (b). In particular, does the car fly off on the same (steeper or gentler) side of the track for the two track types, or on the different sides?

   (c) Set `track_type=1` and find the *minimum* `b` for which there is *no* such initial speed that the car can go over the top and still be on the track at all times. Find the answer with accuracy 0.01 (e.g., $b = 0.25$). Vary the speed with the same accuracy as before, i.e. with increments of ± 0.1.

   (d) Repeat part (c) for `track_type=2`. Compare the answers for parts (c) and (d).

[1] 'p2' in the filename is a legacy of an earlier semester, when this Problem was Problem 2.

2. This problem is worth **2 points**.

Suppose that instead of the equation $y = y(x)$ used for track's shape in the Lecture, we now use parametric equations to describe that shape:

$$x = x(p), \quad y = y(p), \tag{HW4.1}$$

where p is the parameter and $x(p)$ and $y(p)$ are known functions. (You learned about curves described by parametric equations in both Calculus II and Calculus III.)

Let us denote $x' \equiv dx/dp$ and $y' \equiv dy/dp$. Your goal will be to follow the steps presented in the Lecture and obtain counterparts of Eqs. (10b) and (17).

(a) (**0.25 pt**) Use the fact that $d/dx = dp/dx \cdot d/dp$ to show that

$$\frac{d^2y}{dx^2} = \frac{y''x' - y'x''}{(x')^3}\,. \tag{HW4.2}$$

For a later use in this Problem, note that the track has no loops (which we assume to be the case), then

$$x' > 0\,. \tag{HW4.3}$$

(b) (**0.75 pt**) Here you will obtain a counterpart of Eq. (10b) when using Eqs. (HW 4.1) instead of $y = y(x)$.

Recall that when the track is given by equation $y = y(x)$, the convenient variable to describe the motion of the car is x and not the arc length s. When the shape of the track is given by the equations (HW4.1) above, then the convenient variable is p. Therefore, we will seek the equation for $p(t)$ and its derivatives. Moreover, the arc length s is now a function of p:

$$s(p) = \int_0^p \sqrt{\left(\frac{dx}{d\tilde{p}}\right)^2 + \left(\frac{dy}{d\tilde{p}}\right)^2}\, d\tilde{p} \equiv \int_0^p Q(\tilde{p})d\tilde{p}\,. \tag{HW4.4a}$$

This formula defines $Q(p)$. Give a reason from Calculus why $ds/dp = Q(p)$.

Next, follow the derivation of Eq. (11) to obtain:

$$Q' = \frac{x'x'' + y'y''}{Q}\,. \tag{HW4.4b}$$

To complete this part, use the above results and, following the derivation of Eq. (10b), obtain a counterpart of that equation. It should involve dp/dt and d^2p/dt^2 in place of dx/dt and d^2x/dt^2, and possibly some other modifications.

Note: Your answer should reduce to (10b) when $p \equiv x$.

(c) (**1 pt**) Here you will obtain a counterpart of Eq. (17) when using Eq. (HW 4.1) instead of $y = y(x)$.

Recall that the curvature K is defined by Eq. (3) of the Lecture. Follow the lines of the derivation of Eqs. (12) from Eq. (4b) in the Lecture and show that:

$$\frac{d^2x}{ds^2} = \frac{x''Q - x'Q'}{Q^3}, \qquad \frac{d^2y}{ds^2} = \frac{y''Q - y'Q'}{Q^3}, \tag{HW4.5}$$

where $Q' \equiv dQ/dp$. Use the definition of Q^2 to transform Eqs. (HW4.5) to:

$$\frac{d^2x}{ds^2} = \frac{y'(x''y' - x'y'')}{Q^4}, \qquad \frac{d^2y}{ds^2} = \frac{x'(y''x' - y'x'')}{Q^4}\,. \tag{HW4.6}$$

Combine these results with Eq. (3) of the Lecture to obtain:

$$K = \frac{|y''x' - y'x''|}{Q^3}\,. \tag{HW4.7}$$

Finally, use the above results, including those from part (a), and follow the derivation of Eq. (17) to obtain its counterpart, which would involve dp/dt in place of dx/dt, and possibly some other modifications.
Note 1: Your answer should reduce to (17) when $p \equiv x$.
Note 2: You *must explicitly state* where in this derivation you will need the result of part (a) and the assumption (HW4.3). Both of them must be used somewhere!

3. This problem is worth **1.5 points**, with each part contributing half of this weight.

Here, you will return to using the Cartesian equation $y = y(x)$ of the track, as in the Lecture, but consider a situation where the car has been redesigned to move on the bottom side of the track.

(a) Review the calculations presented in the Lecture and determine in which *single* place a change is to be made to distinguish this situation from the one where the car moves on the top side of the track. (This single change will result in modifying *two* numbered equations of the Lecture.) Write down this change, with brief explanations, and also write out these two modified equations.

Is there a certain property of the track's shape[2] for which the car will never be able to stay on its bottom side, no matter what its speed?
Hint: To answer this question, examine the *modified* version of Eq. (17) that you have obtained in this part and determine for what $y(x)$ it can *never* hold true.

(b) Download the codes `hw4_p3.m` and `hw4_p3_track.m` from the Homework webpage. (They differ from the codes for Problem 1 in some details of the track's shape and of the numerical solution of the equations of the motion. The equations themselves are, however, *the same* as in `hw4_p2.m`.)

Introduce the change you made in part (a) into the equations of the motion. Only one quantity will need to be modified (and in a very simple way) in *two* places, one before the loop and one inside the loop.
Write down the command that you modified and the line numbers of its two occurrences in the code.

Run the code `hw4_p3.m` with $b = 0.1$ to find the *minimum* initial speed $V0$ for which the car will be able to stay on the bottom side of the track at all times. Do these simulations for both track types and compare your findings.
Note: For some values of $V0$ you may observe a counterintuitive behavior where the car flies off the track but then goes higher than it. This is a consequence of the shortcoming of the code mentioned in Problem 1, whereby once the car leaves the track, it will never return on it even if it bumps into it at a later time.

For the $V0$ you have found above, answer two more questions:

(i) Is there a particular (steeper or gentler) side of the track from which the car would fly off? Explain why your answer correlates with your last answer in part (a).

(ii) What maximum acceleration (in units of g) would the pilot of the car experience? To answer this question, use the information from the last plot generated by the code `hw4_p3.m`. Recall that 1 Newton = 1 m/s^2 $\cdot$ 1 kg and $g = 9.8$ m/s^2; the mass of the car is stated in the code. You *must use a sketch of the forces* acting on the car to help explain your answer.

See more problems on next page.

[2]Consider only smooth shapes; i.e., no cusps or corners are allowed.

4. This problem is worth **0.25 point**.

   When $|R| = 0$ in Eq. (17), what can you say about d^2x/dt^2 from Eq. (10b)? Now, give an explanation of your answer based on the Second Law of Newton, stated in Lecture 4 by Eq. (1).

Bonus This problem is worth **1.25 points**.
Let us include dry friction into our model. Then Eq. (1) of the Lecture is replaced by

$$m\vec{a} = m\vec{g} + \vec{R} + \vec{F}_{\text{fr}}\,, \qquad \text{(HW4.}B1\text{)}$$

$$\vec{F}_{\text{fr}} = -\mu |R| \vec{T}\,, \qquad \text{(HW4.}B2\text{)}$$

where μ is the friction coefficient and $\vec{T}$ is the unit tangent vector.

(a) Let the shape of the track be given by $y = y(x)$, as in the Lecture, and the track have no loops. Obtain the corresponding generalizations of Eqs. (10b) and (17).

(b) Suppose the car stops on the track (because it had too small an initial speed), and then begins to slide back. How should Eq. (HW4.$B2$) above be modified in this case? Also, propose a formula that would include both your modification and the original (HW4.$B2$). (Recall that the track is assumed to have no loops.)

(c) Modify code `hw4_p2.m` to include friction and redo parts (a) and (b) of Problem 2 with this modified code. Assume $\mu = 0.25$.
Note: You will need to make the same modification in the two lines inside the loop where `VXb` and `VX` are defined.
Strong suggestion: Do *not* make corrections to the original file `hw4_p2.m`. Instead, save its copy into another file and make your corrections in that other file.