MATH 235 — Mathematical Models in Science and Engineering

Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 5

In Problems 2, 3, 5, and 6, make sure to clearly explain how you set up the equations. For examples of clearly explaned setups, see Examples 1–4 in Lecture 5. If you choose to present less detail in your setups, I will choose to reduce your grade.

1. (a) Let P be any $M \times M$ matrix all of whose *row* sums are equal to one (so that P^T is a transition matrix, defined in Eq. (4) in Lecture 5). Verify[1] that it has a steady state vector all of whose entries are equal to $1/M$.
   Hint: The equation satisfied by a steady state vector is listed after the Theorem.
   Note: You should *not* take a matrix P with some specific numeric entries satisfying the description above and then prove the required statement for it. This will *not* be considered a proof. *Rather*, your proof should be for matrix with *symbolic* entries, e.g., p_{11}, p_{12}, etc.. Such a proof for a 3×3 matrix will suffice to illustrate the main idea.

   (b) Show that the transition matrix
   $$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$
   is regular, as defined before Eqs. (16).

   (c) Use the result of part (a) (with an explanation why it applies) to find the steady state vector of matrix P from part (b).

2. John is either happy or sad. If he is happy one day, he is happy the next day four times out of five. If he is sad one day, he is sad the next day one time out of three. Over the long term, what are the chances that John is happy on any given day?
   Note: You should find the answer based on the analysis presented in Examples 1 and 2 in the Notes rather than on a brute force calculation involving many iterations.

3. A country is divided into three demographic regions. It is found that each year, 5% of the residents of region 1 move to region 2 and another 6% move to region 3. Of the residents of region 2, 15% move to region 1 and 10% move to region 3. And of the residents of region 3, 9% move to region 1 and 7% move to region 2. What percentage of the population resides in each region over a long period of time?

[1] Note that *verifying* that something is a solution to an equation is a much easier process than *deriving* a solution.

4. Show that if P is any transition matrix (not necessarily regular), then it has an eigenvalue $\lambda = 1$. Follow these steps.

   1) What are the column sums of $(P - I)$, where I is the identity matrix? What are the row sums of $(P - I)^T$ (the superscript 'T' denotes the transpose of a matrix)?

   2) Use the result of part 1) or of an earlier Problem to show that $(P - I)^T \underline{\mathbf{x}} = \underline{\mathbf{0}}$, where $\underline{\mathbf{x}} = [1\,1\,\ldots\,1]^T$.

   3) What does the result of part 2) tell you about matrix $(P - I)^T$ being singular or nonsingular?

   4) There is a theorem (usually not proved in the first course on Linear Algebra) saying that a matrix A is singular if and only if A^T is singular.[2] Now, finish the proof of the statement made in the first sentence of this problem.

   Note: If you have difficulty with this last part, write down the equation which you are being asked to demonstrate about P.

   Notice that the result you have proved is much less complete than the statement of the Theorem stated after Example 2. For example, it does not guarantee that this eigenvector is unique, no does it say anything about its entries.

5. Three neighbors have backyard vegetable gardens. Neighbor A grows tomatoes, neighbor B grows corn, and neighbor C grows lettuce. They agree to divide their crops among themselves as follows. Neighbor A gets 1/2 of the tomatoes, 1/3 of the corn, and 1/4 of the lettuce. Neighbor B gets 1/3 of the tomatoes, 1/3 of the corn, and 1/4 of the lettuce. Neighbor C gets the remainder. What prices should the neighbors assign to their respective crops if the equilibrium condition of a closed economic model (defined in Example 3) is to be satisfied, and if the lowest-priced crop is to have a price of \$ 100?

6. A town has three main industries: a coal-mining operation, an electric power-generating plant, and a local railroad. To mine \$ 1 of coal, the mining operation must purchase \$ 0.25 of electricity to run its equipment and \$ 0.25 of transportation for its shipping needs. To produce \$ 1 worth of electricity, the generating plant requires \$ 0.65 of coal for fuel, \$ 0.05 of its own electricity to run auxiliary equipment, and \$ 0.05 of transportation. To provide \$ 1 of transportation services, the railroad requires \$ 0.55 of coal for fuel and \$ 0.10 of electricity for its auxiliary equipment. In a certain week, the coal-mining operation receives orders for \$ 50,000 of coal from outside the town, and the power-generating plant receives orders for \$ 25,000 of electricity from outside the town. There is no outside demand for the local railroad. How much must each of the three industries produce in that week to exactly satisfy their own demand and the outside demand?

Bonus This problem is worth **0.5 point**.

In Example 3 in the Notes, assume that a tax agency collects 20% of the earnings of each person as tax. Present a modified analysis of this problem and conclude whether a new choice of prices p_1, p_2, p_3 exists. If it does, find it.

Note: As before, you must clearly explain your setup. Without such an explanation, I will not even consider your solution.

[2] You could have seen this theorem before in different (but equivalent) forms, e.g.: (i) the number of linearly independent columns in a matrix equals the number of its linearly independent rows; or (ii) det(A) =det(A^T), plus the fact that the determinant of a singular matrix is zero. You will *not* need to use these other forms here.