MATH 235 — Mathematical Models in Science and Engineering

Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 6

1. In a Calculus book (or elsewhere), find the Taylor formula with a remainder.[1] Your goal in this problem is to use that formula and estimate the maximum angle θ for which the approximation $\sin\theta \approx \theta$ has an error of at most 10%. To that end, proceed in two steps.

   First, decide what degree n to seek in this remainder term. Justify your answer. The accuracy of your approximation will significantly depend on your answer in this step.

   Second, apply the remainder formula to find the maximum value θ satisfying the condition for the error described above. Express this angle in degrees.

   Conclude by stating for what amplitude of the pendulum's oscillations the harmonic oscillator equation $\ddot{\theta} = -\theta$ is a decent (i.e. within 10% or so) approximation of the pendulum's motion.

   Note: When computing the error with the exact value of $\sin\theta$, it is appropriate to replace it with its approximate value, i.e., θ.

2. This problem is worth **0.5 point**.

   (a) Using (9a), derive (9b).

   (b) Using Eqs. (9), verify that
   $$e^{i\pi/2} = i, \qquad e^{-i\pi/2} = -i\,.$$

   (c) Using Eqs. (9) and (10a), express $\cos\tau$ and $\sin\tau$ as linear combinations of $e^{i\tau}$ and $e^{-i\tau}$.

   (d) Let $z = a + ib$ and $w = c + id$ be any two complex numbers (where a, b, c, d are real). Verify that
   $$\overline{z\,w} = \overline{z}\,\overline{w}\,.$$

3. This problem is worth **1.25 points**.

   In this problem you will explore the behavior of a solution that contains terms evolving over disparately different time scales (fast and slow).

   (a) Consider an oscillator with a very large friction ($\gamma \gg 1$) and no external force. Using the Hint below, find the main-order approximations in this limit for the characteristic roots λ_1 and λ_2. That is, keep only the largest nonzero term in the expression for each of λ_1 and λ_2.

   Thus, the general motion of a highly damped oscillator is described by a linear combination of two exponentials, $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. Plot these exponentials versus t and, based on this, answer two questions:

[1] Just in case, to address an issue that has come up in the past: There should be no integral in this formula.

(i) Use an adjective to describe the behavior of each of these exponentials. (If you are not sure what this is asking for, re-read the introductory sentence to this problem.) Conclude which exponential will govern which part of the motion (i.e., for short or for long times) of a highly damped oscillator.

(ii) Which exponential will govern the behavior of the solution for sufficiently large times? (Such a behavior is called the *asymptotic* behavior.)

Hint: To find the approximation of $\sqrt{\gamma^2 - 1}$, use the following common trick:

$$\sqrt{\gamma^2 - 1} = \gamma \sqrt{1 - \frac{1}{\gamma^2}},$$

and now use the binomial expansion to obtain the first two terms of the above expression.

(b) Using the results you obtained in part (a), find the constants c_1 and c_2 in (17a) such that $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \Omega_0$.
Note: Use *Mathematica* to solve the linear system. (As always, include your *Mathematica* work in your document submitted for grading.)

(c) Plot the resulting $\theta(\tau)$ for $\gamma = 5$, $0 \leq \tau \leq 10$, and two sets of initial conditions:
(i) $\theta_0 = (\pi/4), \quad \Omega_0 = 0$;
(ii) $\theta_0 = 0, \quad \Omega_0 = 2\gamma\,(\pi/4)$.
Label (by hand is okay) the regions where the solution's behavior is governed mainly by only one of the exponentials, i.e. by either $e^{\lambda_1 t}$ or by $e^{\lambda_2 t}$. Also label the type of behavior, according to your answer to question (i) in part (a), for each region.

The behavior that you should observe in one of these plots is referred to as a "boundary layer". In a boundary layer, Nature employs the fast solution to match the initial condition with the slow solution, which dominates the evolution of θ over most of the time interval.
Which of the cases, (i) or (ii), has a boundary layer?
Can you see a mathematical reason why the other case does not have a boundary layer?
Hint: Look at the expression for one of the constants, c_1 or c_2.

Technical note: For plotting purposes, it is convenient to use *Mathematica* and define θ to be a function of t, γ, θ_0, and Ω_0.

4. It is easy to see how the initial condition $\theta(0) = \theta_0$ can be prepared: one simply grabs the pendulum and pulls it by a prescribed amount from the equilibrium. In this problem you will explore how the *other* initial condition, $\dot{\theta}(0) = \Omega_0$, can be prepared.

Consider the effect of striking the pendulum that is initially at rest. The problem is to solve

$$\ddot{\theta} + 2\gamma\dot{\theta} + \theta = \mathbf{f}(\tau) \qquad \text{(HW6.1)}$$

(where now *no* assumption is made about the size of γ), subject to the initial condition

$$\theta(0) = \theta_0, \qquad \dot{\theta}(0) = 0\,. \qquad \text{(HW6.2)}$$

Assume that the force due to the striking, $\mathbf{f}(\tau)$, is approximately constant for some short interval of time $\Delta\tau \ll 1$, and then zero thereafter:

$$\mathbf{f}(\tau) = \begin{cases} \mathbf{f}_0, & 0 \leq \tau \leq \Delta\tau \\ 0, & \tau > \Delta\tau\,. \end{cases} \qquad \text{(HW6.3)}$$

Further assume that the force is large during the time of the strike, i.e. $\mathbf{f}_0 \gg 1$, in such a way that the *product* $\mathbf{f}_0 \Delta\tau \equiv \Omega_0$ (called the *impulse*) is finite, i.e., $\Omega_0 = O(1)$.

Your goal is to show that at the end of the strike,

$$\theta(\Delta\tau) = \theta_0 + O(\Delta\tau), \qquad \dot{\theta}(\Delta\tau) = \Omega_0 + O(\Delta\tau)\,. \tag{HW6.4}$$

Then, taking the limit of $\Delta\tau \to 0$ in Eqs. (HW6.4) one recovers the desired initial conditions with a nonzero initial velocity:

$$\theta(0) = \theta_0, \qquad \dot{\theta}(0) = \Omega_0\,. \tag{HW6.5}$$

(a) Starting with (HW6.1) and using the Fundamental Theorem of Calculus, show that for $0 \le \tau \le \Delta\tau$,

$$\dot{\theta} + 2\gamma(\theta - \theta_0) + \int_0^\tau \theta(\tilde{\tau})d\tilde{\tau} = \Omega_0 \frac{\tau}{\Delta\tau}\,. \tag{HW6.6}$$

Note that the term on the r.h.s. is $O(1)$ rather than $O(\tau)$ because $\tau = O(\Delta\tau)$.

You will now need to use Eq. (HW6.6) to obtain the first equation in (HW6.4); the second equation there will be obtained in part (b). Your first step is to estimate the order of magnitude of the second and third terms on the l.h.s. of (HW6.6). To that end, *assume* that

$$\theta(\tau) = \theta_0 + O(\tau^\alpha) \qquad \text{for some } \alpha > 0. \tag{HW6.7}$$

Let us explain why one should have $\alpha > 0$. Indeed, since $\lim_{\tau\to 0} O(\tau^\alpha) = 0$ for $\alpha > 0$, then (HW6.7) means that at the instant immediately after the strike, the pendulum is still close to its initial location — a reasonable assumption indeed. On the other hand, if we had assumed $\alpha = 0$, then (HW6.7) would imply that in the instant immediately after the strike, the pendulum's bob gets "teleported" to a location $\theta_0 + O(1)$, since $O(\tau^0) = O(1)$. This does not sound reasonable, and thus we have excluded the case $\alpha = 0$ from assumption (HW6.7). You should verify on your own that the assumption $\alpha < 0$ would not have made sense, either.

Now, to obtain the first equation in (HW6.4), rewrite (HW6.6) as:

$$\dot{\theta} = -2\gamma(\theta - \theta_0) - \int_0^\tau \theta(\tilde{\tau})d\tilde{\tau} \;+\; \Omega_0 \frac{\tau}{\Delta\tau} \tag{HW6.6$'$}$$

and integrate it one more time. In so doing, you need to substitute (HW6.7) into the *r.h.s. only* of (HW6.6$'$).

To conclude this step, explain why your last result implies that α *must* equal 1 in (HW6.7). Now state the main result that you have obtained in this part. If you are unsure what it is, re-read the text after (HW6.6).

(b) It remains to obtain the second equation in (HW6.4). To this end, simply substitute (HW6.7) with $\alpha = 1$ into (HW6.6$'$) and verify that it yields the desired initial condition.

5. Use limit (b) ($f \to 1,\ \gamma = 0$; see Sec. 6.9) and follow lines similar to the derivation of Eq. (24) in the Notes to re-derive that equation.

6. This problem is worth **1.25 points**.

   Here you will show that the harmonic oscillator equation (4) arises in the description of the bobbing motion of a floating cylinder. Consider a cylinder of uniform mass density ρ, having mass m and cross-sectional area A. Assume that it is floating in a liquid having density ρ_l, where $\rho < \rho_l$.

(a) Two forces act on the cylinder: the gravity and the buoyancy force. As discovered by Archimedes, the buoyancy force equals the weight of the liquid displaced by the part of the body immersed into it. In the equilibrium state, these two forces balance each other when the cylinder sinks into the liquid by some depth $|Y_E|$, where $Y_E < 0$ is the y-coordinate of the bottom of the cylinder, with the liquid's surface being at $y = 0$. Find this Y_E.

(b) Now suppose that one displaces the cylinder from this equilibrium by either pulling it up or pushing down. Then the coordinate of its bottom becomes $Y = Y_E + y$, where y is the displacement from the equilibrium. Use the Second Newton's Law to show that y satisfies an equation of the form

$$\frac{d^2y}{dt^2} = -\omega^2 \, y \, . \tag{HW6.9}$$

Find the ω, which is called the frequency of the oscillations.

Note: You must sketch a picture of the partially submersed cylinder and also the force diagram to be able to set up and solve this problem correctly.

(c) Suppose one has two cylinders made of the same material and having the same mass but different cross-sectional areas. Which cylinder will oscillate more rapidly, the shorter one or the taller one? (The higher the frequency of the oscillations, the more rapidly it oscillates.) Justify your answer.

(d) Same questions about two cylinders of the same dimensions but made of different materials (a heavier one and a lighter one).

Bonus (worth **0.5 point**) Using the complex representation of an exponential (see, e.g., (17b)), evaluate

$$\int e^{ax} \sin(bx) \, dx \, ,$$

where a and b are some real coefficients.

No credit will be given for finding the answer with any other method.