Sec. 11.11 Approximating functions by their Taylor polynomials.

① Approximations based on Taylor's Remainder formula

We have seen that the Taylor polynomial approxima. $f(x)$ near $x=a$. The more terms we take in the polynomial, the better ~~is~~ this approximation. In general, we can say

$$f(x) = p_n(x) + R_n(x)$$

function — nth Taylor Polynomial — Remainder.

To know the accuracy of the approximation of $f(x)$ with $p_n(x)$, we need to have an estimate for $R_n(x)$.

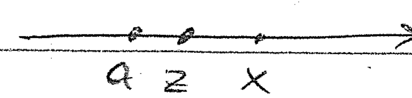
Thm. 11.10.7 Taylor's (Remainder) formula

If $f(x)$ has $(n+1)$ derivatives ~~in an interval~~ at ~~containing~~ $x=a$ and nearby points, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

and z is some point between x and a.



Note: In most cases one cannot find this z, but still can use the above formula to estimate the Remainder by finding the upper bound for $R_n(x)$.

Ex. 1 Use the Taylor's Remainder Formula to estimate the range of values for x for which the approximation

$$\cos x \approx 1 - \frac{x^2}{2}$$

is accurate to within 0.001.

(a) Coarse estimate

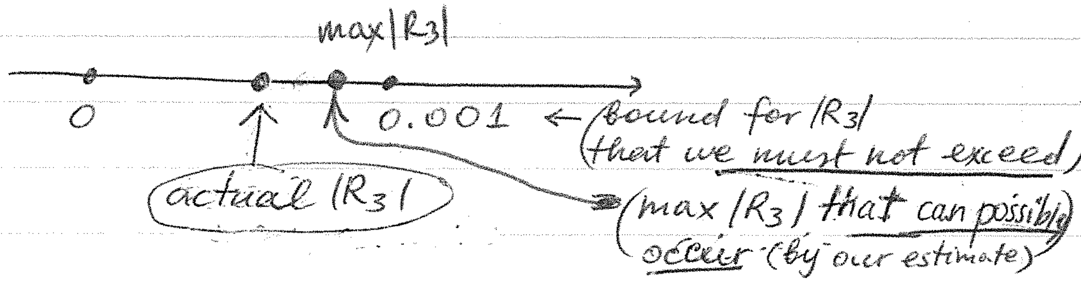
1) Remainder formula:

$$\cos x = 1 - \frac{x^2}{2} + R_3(x)$$

$$R_3(x) = \frac{\cos^{(3)}(z)}{3!} x^3 = \frac{-\sin z}{3!} x^3, \quad 0 < z < x. \quad (\text{if } x<0,\ x < z < 0.)$$

2) $|R_3(x)| \le 0.001 \Rightarrow$

$$\frac{|\sin z|}{3!} x^3 \le 0.001$$



$$\max |R_3(x)| \le \max_z (|\sin(z)|) \cdot \frac{x^3}{3!} = 1 \cdot \frac{x^3}{3!}$$

So, we will satisfy $|R_3(x)| \le 0.001$ if we replace $R_3(x)$ by its max. possible value $1 \cdot \frac{x^3}{3!}$.

Then $\frac{x^3}{3!} \le 0.001 \Rightarrow x^3 \le 6 \cdot 0.001 \Rightarrow$

$x \le 0.182_{rad} = 10.4^\circ$.

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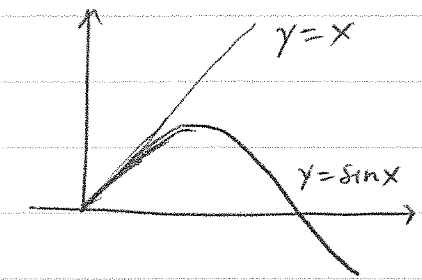
(b) Finer estimate

$\max|R_3(x)| \leq \max\limits_{0<z<x} |\sin z| \cdot \frac{x^3}{3!}$

the difference from (a).



$|\sin z| \leq x$,

$\Rightarrow \max|R_3(x)| \leq x \cdot \frac{x^3}{3!} = \frac{x^4}{3!}$.

Then (as before): $\frac{x^4}{3!} \leq 0.001 \xRightarrow{\max|R_3| \leq 0.001} x \leq 0.28\,\text{rad} \approx 15.9^\circ$.

Thus, by using a more careful estimate for $\max|R_3(x)|$, we obtained a large interval for allowed x.

Ex. 2 Use the Taylor's Remainder Formula to estimate the max error of $\cos x \approx 1 - \frac{x^2}{2}$ when $0 < x < \pi/6$ $(=30^\circ)$.

Ex. 1: Find max(x) given max possible error.
Ex. 2: Find max possible error given max(x).

Sol'n ~~using the finer estimate~~.

$|R_3(x)| \leq \max\limits_{0<z<\pi/6} |\sin(z)| \cdot \max\limits_{0<x\leq\frac{\pi}{6}} \frac{x^3}{3!}$

$= \sin\frac{\pi}{6} \cdot \frac{(\pi/6)^3}{3!} = \frac{1}{2} \cdot \frac{(\pi/6)^3}{6} = 0.012$.

Thus, $|R_3(x)| \leq 0.012$.

Check: $\cos(\pi/6) = \sqrt{3}/2 = 0.8660$

$1 - \frac{(\pi/6)^2}{2} = 0.8629$

The difference is actually ≈ 0.003, i.e. our estimate is quite conservative.

Must See also Ex. 1 in book; Ex. 2 is helpful but optional.
(approx $\sqrt[3]{x}$ around $x=8$)

② Estimate using the Alternating Series Remainder.

Ex. 3 Repeat Ex. 1 using the Alt. Series Remainder formula (Sec. 11.5): I.e., find max(x) for which the max error of $(1-\frac{x^2}{2})$ will be ≤ 0.001.

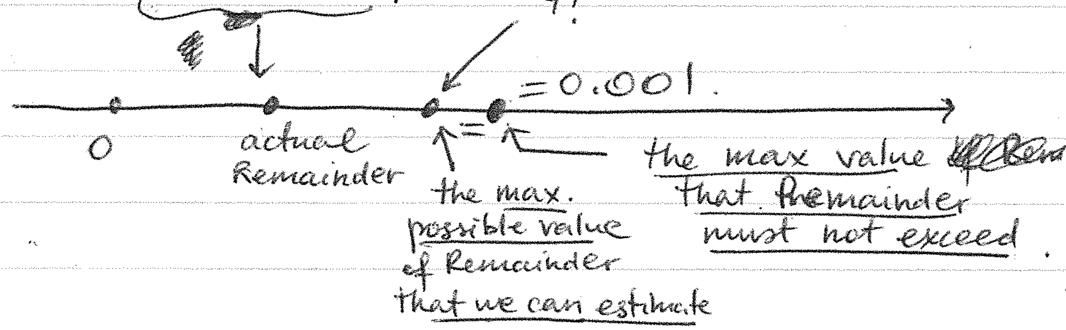
Sol'n: We know that the Maclaurin series for $\cos x$ is an Alternating series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

From Sec. 11.5 we recall:

$$|\underset{\substack{\uparrow\\ \text{2nd part. sum of the series}}}{S_2} - \cos x| \le |\underset{\substack{\uparrow\\ \text{3rd term}}}{a_3}|$$

$$\left|1 - \frac{x^2}{2} - \cos x\right| \le \frac{x^4}{4!}$$



Thus, we require $\frac{x^4}{4!} \le 0.001 \Rightarrow x^4 \le 24 \cdot 0.001$

(vs. $x^4 \le 6 \cdot 0.001$ in Ex. 1(b).)

$x \le 0.39$ rad $\approx$ **22.6°** (vs. 0.28 rad = 15.9° in Ex. 1(b)).

Note: We obtain this better estimate using the AS Remainder Formula because we used the additional info that the series for $\cos x$ is alternating.

(The Taylor Remainder Formula doesn't tell us that the series is alternating.)