Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

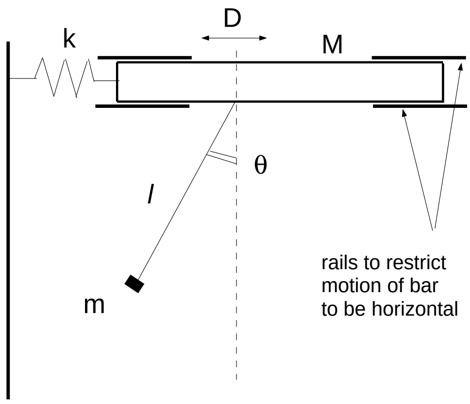
Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 7

Problems 1, 2, and 3 below are similar to the examples considered in Sections 7.3, 7.4, and 7.5. When you work on these problems, please mimic your steps after the respective steps found in the notes. If you notice that you have arrived at an equation that is the same (or equivalent, after renaming a constant) to an equation considered in Sections 7.3 – 7.5, you may simply quote the results for that equation without rederiving it, *unless the problem specifically instructs otherwise*.

Problems 1, 2, and 3 are worth 3, 1, and 2 points. This reflects the amount of work that I expect you to do for each problem. **This note is intended to help you avoid doing some unnecessary work.**

1.



(a) (**0.75 pts**) Consider a horizontal bar of mass M attached to a wall by a spring with the spring constant k, as shown in the figure on the left. The bar is restricted to move only in the horizontal direction. A simple pendulum with a weightless rod of length l and a point mass m at the end is attached to the bar and performs swinging motion. Use the Euler–Lagrange equations to derive the equations of motion for such a system. Your equations should be for θ and D, where D is the amount of compression or stretching of the spring beyond its natural length.

(b) (**0.75 pts**) Recall that whenever you obtain an answer, it is always a good idea to test its special cases to see if they agree with previously known answers. In this spirit, test the equations you have obtained as follows.

(i) In the Euler–Lagrange equation obtained in part (a) by differentiation with respect to θ and $\dot{\theta}$, set D and all its derivatives to zero. You should then recover the equation for the simple pendulum.

(ii) Likewise, in the equation obtained by differentiation with respect to D and $\dot{D}$, set θ and all its derivatives to zero and then verify that you obtain the equation of a mass on a spring.

(iii) Verify that in the limit of a very heavy bar, one of the two Euler–Lagrange equations reduces to the equation of a simple pendulum.
Hint: Show that one of the Euler–Lagrange equations implies that for $M \gg 1$, one has to have $\ddot{D} = O(1/M) \ll 1$. Take the limit $M \to \infty$, and then deduce the conclusion required above.

Side note 1 (JFYI) Notice that in your argument for case (iii), one has to be careful when taking the limit of $M \to \infty$. Namely, if one merely sets $\ddot{D} = D = 0$ in both Euler–Lagrange equations, one can see that θ has to satisfy two equations which contradict each other. This conundrum disappears if, instead, one keeps into account that $\ddot{D} = O(1/M)$ and therefore a term $O(M) \cdot \ddot{D}$, which appears in one of these equations, is of order one, i.e. of the same size as the θ-terms.

It is also interesting to note that in the limit of a very tight spring, the equations do *not* reduce to the equation of a simple pendulum. Indeed, in this case, it is straightforward to see that D is very small, but this does *not* imply that $\ddot{D}$ is small (because D oscillates with very high frequency). A more subtle analysis, which is far beyond the scope of this course, is required to to handle this case.

(c) (**1.5 pts**) Derive the linearized equations for small oscillations near the equilibrium state(s) of this system and then perform the stability analysis of these states.

Guidelines for the stability analysis
Let the small deviations of θ and D from their respective equilibrium values be φ and δ. The linearized equations that you are supposed to obtain in part (c) can be written in matrix form as

$$A\ddot{\underline{\mathbf{v}}} = B\underline{\mathbf{v}}, \qquad \underline{\mathbf{v}}(t) = \begin{pmatrix} \varphi \\ \delta \end{pmatrix}, \tag{HW7.1}$$

where A and B are some 2×2 matrices whose explicit form you will determine. As a reference, A should have all of its entries nonzero, while B should be diagonal. Observe that this is a matrix analog of the scalar harmonic oscillation model $l\ddot{\varphi} = -g\varphi$. Therefore, the method of solution will follow similar lines. Namely, substitute into (HW7.1)

$$\underline{\mathbf{v}}(t) = e^{\lambda t}\underline{\mathbf{u}}, \tag{HW7.2}$$

where $\underline{\mathbf{u}}$ is a *constant* (i.e., t-independent) vector. The result of this substitution is (verify and show your work):

$$(\lambda^2 A - B)\underline{\mathbf{u}} = \underline{\mathbf{0}}\,. \tag{HW7.3}$$

This says that substitution (HW7.2) does indeed give a solution of (HW7.1) for such values λ that make matrix $(\lambda^2 A - B)$ singular (*explain why*). The rest is similar to finding eigenvalues and eigenvectors of a matrix. (In fact, (HW7.3) is sometimes referred to as a generalized or a quadratic eigenvalue problem.) Namely, find the λ^2's such that $\det(\lambda^2 A - B) = 0$. If all (in this problem, both) λ^2's are negative, then the equilibrium is stable. In order to receive full credit, you *must explain why this is so, following the stability analysis in Lecture 6*. If any of the λ^2's is positive, then the equilibrium is unstable (again, explain why, also following Lecture 6).

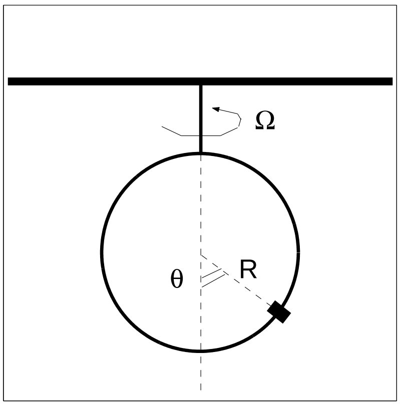
Technical note You may obtain expressions for λ^2's by hand or with *Mathematica*. These expressions should have the form $(\pm\sqrt{a^2 + b} + c)$ for some a, b, c. (Some of these coefficients may be negative.) You should determine the sign of those expressions *in general*, i.e. without assuming any specific values for m, M, g, etc.. Note that as a first step of that explanation, you *must show* that the expression under the radical is positive: $a^2 + b > 0$. Indeed, if this had not been so, then λ^2's would have been complex.
Hint for showing that $a^2 + b > 0$: First explain why $(p + q)^2 - 4pq \geq 0$ for any p and q.

Side note 2 (JFYI) To each of the λ^2's there corresponds its vector $\underline{\mathbf{u}}$ (similarly to how to an eigenvalue of a matrix there corresponds an eigenvector). This vector is called a *normal mode*, or an *eigenmode*, of Eq. (HW7.1). If $\lambda_1^2,\ \underline{\mathbf{u}}_1$ and $\lambda_2^2,\ \underline{\mathbf{u}}_2$ are two different solutions of (HW7.3), then by the linear superposition principle, an arbitrary solution of (HW7.1) has the form

$$\underline{\mathbf{v}}(t) = \left(c_1\, e^{\lambda_1 t} + c_2\, e^{-\lambda_1 t}\right)\underline{\mathbf{u}}_1 + \left(c_3\, e^{\lambda_2 t} + c_4\, e^{-\lambda_2 t}\right)\underline{\mathbf{u}}_2, \qquad \text{(HW7.4)}$$

where the c's are determined by initial conditions. Thus, *any motion of a linear system can be represented as a superposition of its eigenmodes*. This observation illustrates the significance of eigenmodes.

Side note 3 (JFYI) The model considered above is a piece of a baby model for the problem of synchronization of two pendula attached to a common support. In the 17th century, the great Dutch researcher Christian Huijgens (Huygens) discovered that two clocks (at that time, basically, pendula) hanging on the same wall would synchronize, even if originally they had slightly different frequencies and arbitrary phases. The phenomenon of synchronization was extensively studied in the 20th century and has applications in diverse areas of life.

2. 

(a) Consider a circular wire hoop of radius R suspended from the ceiling, as shown on the left. Suppose the hoop is rotating about its vertical axis with a constant angular velocity Ω, and a bead is able to slide along the wire without friction. Use the Euler–Lagrange equations to obtain the equation of motion for the angle θ indicated in the figure.

(b) Derive the linearized equation for small oscillations near the equilibrium states of this system and then perform the stability analysis of these states. Draw a bifurcation diagram.

3. (a) Consider a simple pendulum whose pivot point is rapidly oscillating in the *horizontal* direction, with the coordinate of the pivot point being $x_0(t) = a\,\cos(\Omega t)$. The rest of the assumptions are the same as for a similar model considered in Section 7.5 of the Notes. In particular, we assume that $R \equiv \Omega/\omega_0 \gg 1$, where $\omega_0^2 = g/l$ and l is the length of the pendulum, and $\epsilon \equiv a/l \ll 1$. Use the Euler–Lagrange equations to derive the *exact* equation of motion for the angle θ of such a pendulum. (This equation is an analogue of either of Eqs. (22) in the Notes.)

   (b) Represent the solution of this equation as a sum of the fast and slower solutions, as in Eq. (24) of the Notes, and then derive an equation for the slower solution $\theta^{(0)}$. This should be an analogue of Eq. (28) in the Notes.

   (c) Derive a linearized equation for small oscillations near the equilibrium states of this system and then perform the stability analysis of these states. Draw a bifurcation diagram.