MATH 235 — Mathematical Models in Science and Engineering

Instructions: Present your work in a neat and organized manner. Please **use** either the 8.5×11 size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used *Mathematica* for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

Homework Assignment # 8

1. This problem is worth **0.5 point**.

   (a) Use the method employed in the derivation of Eqs. (7) and (26) to derive a trigonometric identity representing
   $$\sin x\, \sin y$$
   as a sum or difference of certain sine and/or cosine functions.

   (b) Same for
   $$\sin x\, \cos y\,.$$

2. This problem is worth **0.5 point**.

   Plot together $\cos(t)$ and $\cos(1.1t)$ on such an interval $[0, T]$ that these two functions differ by a number of order one (e.g., some number between 0.5 and 2) at the right end of the plot.
   How big, in the order of magnitude sense, is $0.1T$ for this T?

   Explain how this part is related to the discussions found after Eqs. (11) and (20).

3. This problem is worth **2 points**.

   (a) (**0.25 pt**) Consider a sphere centered at the origin and having the radius R. Use the method of disks (from Calculus II) to find the volume of the part of this sphere included between its lowest point and a horizontal plane $z = G$ (for $-R < G < R$, of course).
   Suggestion: After you will have obtained an answer, it is a good idea to verify that it yields familiar results in certain special cases.

   (b) (**1 pt**) **Before you attempt this problem, review your solution of Problem 6 of HW 6, as the setups of that and the present problems are very similar.**

   (i) Consider a ball of uniform density ρ and radius R that is floating in a liquid of density $\rho_l > \rho$. Let the liquid's surface be at $z = 0$ and the ball's center be at $z = H$. Draw a picture to relate H to the G from part (a). Now, using your result from part (a), write down the expression for the volume of the submerged part of the ball.

   Suggestion 1: Before you do this part, re-read how G is defined above. Now, to help yourself to relate G and H, draw the ball (well, its side view) and two parallel z-axes (say, blue and red). On the blue axis, mark points $z = -R$, G, and 0 *as defined in part (a)*. On the red one, mark $z = 0$ and $z = H$ *as defined in this part*.

   Suggestion 2: As the sanity check, verify that values $H = -R,\ 0,\ R$ in your formula yield reasonable values for the submerged volume.

(ii) Denote H_E to be the value of H occurring when the ball is in equilibrium. From the balance of the gravity and buoyancy forces, write an equation satisfied by H_E. (Do **not** attempt to solve this cubic equation!)
Next, denote a ratio $r = H_E/R$ and use your equation for H_E to obtain an equation for r. This is still a cubic equation for r, so you should *not* attempt to solve it. However, it clearly shows a *single* parameter on which this ratio depends. State what this parameter is. Also, write a brief paragraph about whether your intuition agrees with the fact that there is only one parameter that determines r, or you think there may be other parameters that r can depend on.

(iii) Now consider a deviation h of the ball from the equilibrium, whereby $H = H_E + h$. Derive an exact equation for h using a method analogous to that used in Problem 6 of HW 6. This should be a harmonic oscillator equation plus some nonlinear terms.

(c) (**0.75 pt**) Inspect the coefficient of the *linear* term in that equation and answer the following questions, *providing sufficient explanations and necessary calculations*.

(i) Given two balls made of the same material but having different radii, which of the balls will oscillate faster?

(ii) Same question about two balls of the same radius but made of different materials, both satisfying the condition $(\rho_l/2) < \rho < \rho_l$.[1] This condition means that the balls are submerged by more than half.

Note 1: The coefficient in question involves a fraction of the form $(a^2 - b^2)/a^3$, which can be rewritten as follows:

$$\frac{a^2 - b^2}{a^3} = \frac{1}{a} \cdot \left(1 - \left(\frac{b}{a}\right)^2\right).$$

Use this identity and a result from part (b)(ii) to answer question (c)(i).
Note 2: To answer question (c)(ii), you would ideally need the solution of a cubic equation mentioned in part (b)(ii). However, you do not know that solution. Nonetheless, when the ball is submerged more than half, you *can* use your intuition about how deeply a ball would sink in the liquid depending on its density. Use this observation and also Note 1 to answer question (c)(ii).

4. This problem is worth **1.5 points**.
In this exercise your goal will be to explain *qualitatively* what determines whether the frequency of the oscillations increases or decreases with their amplitude.

Note: This assignment is unusual in that it does not require you to do any calculations. Instead, it emphasizes a qualitative understanding of the phenomenon. Thus, *your goal is to present your answer in a coherent and clear manner, focusing on the logic of your exposition* rather than on mathematical details. *I will reduce your score* when grading this problem if I am unable to follow, without undue hardship, your explanations, even if they are correct.

Preamble: Consider the equation for the mass on a spring: $m\ddot{x} = -kx$. The tighter the spring that provides the restoring force, and hence the higher the spring constant k, the higher the frequency of the oscillations, $\sqrt{k/m}$. More generally, consider an equation $\ddot{x} = -f(x)$ where $f(x)$ is an approximately linear function: $f(x) \approx x\cdot$(a slowly varying function of x). Then, a conclusion similar to the one above will hold, where now the role of the spring constant will be played by the "slowly varying function of x" in the previous sentence. This slowly varying function can also be interpreted as the restoring force "per unit change in length x".

[1] See Bonus # 1 for the other case.

(a) Now apply this idea to a pendulum whose oscillations have a small but finite amplitude. Look at Eq. (2). When the $O(\varphi^3)$ term is neglected, the restoring force "per unit change in φ" is proportional to $\omega_0^2 = g/l$. Now, when the $O(\varphi^3)$ term is returned into the picture, how does this affect the restoring force "per unit change in φ"? Using the analogy with the mass on a spring, predict how this would in turn affect the frequency of the oscillations. In other words, according to (2), does the frequency increase or decrease with the amplitude? Is this what Eq. (23) predicts?

(b) Consider oscillations of a floating ball from Problem 3 above. Assume that $\rho = \frac{1}{2}\rho_l$, so that $H_E = 0$. By *drawing an accurate picture* (or pictures) and *without doing any calculations*, explain in what way the restoring force on the ball is different from the restoring force on a vertical cylinder with the same radius and mass as the ball. (In other words, what is the principal, *qualitative* difference of the restoring forces per unit change in depth in these two cases?) Based on this explanation, predict whether the frequency of the ball's oscillations will increase or decrease with the oscillations' amplitude. Finally, display the equation for h in this case (what is r?) from Problem 3(b) and argue why your prediction about the frequency as made above is consistent with that predicted by this equation.

Hint: The restoring force is the excess of the buoyancy force over gravity. (This excess amount can, by definition, be positive or negative.) This excess buoyancy is proportional to the *excess* submerged volume when the ball goes either slightly below or above the equilibrium. Your drawing must show this excess volume as a flat disk with a non-constant cross-section; in particular, you must *carefully* depict the curvature of this disk.

(c) Without performing any calculations, sketch a solid for which the frequency of oscillations in a liquid would have the opposite dependence on the amplitude than it does for the ball. Explain your reasoning based on your answers to parts (a) and (b).

Bonus # 1 (**0.5 pt**) Answer question (ii) in Problem 3(c) for the case opposite of the one considered there, i.e. for $\rho < (\rho_l/2)$.
Hint: Graph the coefficient in question in such a way that you could infer how this coefficient changes depending on the parameter stated in the question.

Bonus # 2 (**1 pt**) Consider a solid of revolution with uniform density ρ. The solid is floating in a liquid of density $\rho_l > \rho$ so that its axis of revolution is vertical. Suppose that when in the equilibrium, the solid is submerged into the liquid up to a mark z_0 on its vertical axis. Furthemore, suppose that the radius of the solid near that mark can be approximated as

$$r(z) = r_0 + r_1(z - z_0) + \frac{1}{2}r_2(z - z_0)^2\,,$$

where the z-axis is assumed to point *upward*. (This r has no relation to that introduced in Problem 3(b).)

Derive an equation for small but finite oscillations of such a solid keeping terms up to $(z - z_0)^3$ in this equation. Your derivation must *clearly* show how certain forces cancel in the equilibrium.

Answer the following questions:

(i) Under what conditions on r_0, r_1, and r_2 will that equation *not* have the quadratic term? Draw an *inambiguous* picture of the solid of the corresponding shape.

(ii) Same question about the cubic term.

(iii) Within the above model for $r(z)$, is there a case (other than $r_1 = r_2 = 0$) where both the quadratic and cubic terms vanish? If so, draw a picture of such a solid.

(iv) Consider the case where the cubic term is absent, but the quadratic term may be nonzero. Based on the material of the Lecture, explain what effect(s) on the oscillations the quadratic term will have. That is, how is the motion in this case different from that in the purely linear case (i.e. when $r_1 = r_2 = 0$)?