## MATH 235 — Mathematical Models in Science and Engineering

**Instructions**: Present your work in a neat and organized manner. Please **use** either the $8.5 \times 11$ size paper or the filler paper with pre-punched holes. Please do **not use** paper which has been torn from a spiral notebook. Please secure all your papers by using either a staple or a paper clip, but **not** by folding its (upper left) corner.

You **must** show **all of the essential details** of your work to get full credit. If you used Mathematica for some of your calculations, attach a printout showing your commands and their output. If I am forced to fill in gaps in your solution by using notrivial (at my discretion) steps, I will also be forced to reduce your score.

Please refer to the syllabus for the instructions on working on homework assignments with other students and on submitting **your own** work.

# Homework Assignment # 9

1. This problem is worth **1.5 points**; three parts are *not* related to one another.
   You may find solutions of parts (a) and (b) online or in textbooks. In either case, you *must* cite your sources. Also, you must show *all steps* of your derivations, even those omitted in your sources.

   (a) Derive Eq. (2) of the Notes.

   (b) Use either L'Hôpital's Rule or a technique based on the Maclaurin expansion of $\log(1+x)$ (see the formula in Problem 1 of HW 3) to derive Eq. (20) of Lecture 9.

   (c) In the Notes we showed that for $\omega \gg 1$,

   $$\int_0^T \sin(\omega t)\, dt = O\left(\frac{1}{\omega}\right),$$

   with the maximum value for this integral being achieved when the interval $[0,\, T]$ contains a semi-integer number of periods $2\pi/\omega$. Here you will demonstrate that the same estimate holds for

   $$\int_0^T f(t)\, \sin(\omega t)\, dt,$$

   where $f(t)$ changes slowly compared to $\sin(\omega t)$.

   Use Mathematica to verify that the following integrals are $O(1/n)$ for $n \gg 1$. For simplicity, assume that $n$ is an <u>odd</u> integer.

   $$\int_0^\pi t\, \sin(n\, t)\, dt, \qquad \int_0^\pi t^2\, \sin(n\, t)\, dt, \qquad \int_0^\pi e^t\, \sin(n\, t)\, dt.$$

   Find the numerical coefficient in front of the $1/n$ term in each case. See the Hint below.
   *Hint 1*: Find the leading-order terms in the numerator and denominator of your answer. The technique for doing so was covered in various parts of Calculus, and so you may need to review your Calculus textbook. You first encountered this technique in Calculus I when finding the limit of a fraction of two polynomials of the form $\infty/\infty$. (This was before you studied derivatives, i.e. well before you learned L'Hôpital's Rule.) You also used it in Calculus II when you learned the Limit Comparison Test for series.
   *Hint 2*: What are $\cos \pi n$ and $\sin \pi n$ when $n$ is an odd integer?

2. This problem is borrowed from Sec. 4.1 of the book by C. Groetsch "Inverse Problems".

   In Lecture 9 we learned how, given an initial concentration of a substance in a reservoir and also the in- and outflow rates to and from the reservoir, one can find the concentration in the reservoir at all times. Below you will be asked to solve an *inverse problem*.

   Groundwater having an unknown but constant concentration of pollutants seeps at an unknown but constant rate into a cistern of volume $V_0$. The well-stirred mixture leaks out at the same rate. The initial concentration of pollutants is 1%. After 1 day the concentration of pollutants is 1.1%, and after 2 days it is 1.19%. What is the concentration of pollutants in the groundwater and at what rate is the groundwater seeping into the cistern?

*Note*: You must solve the equations for $r$ and $c_{\text{in}}$ by hand, *not* with Mathematica.
*Hint*: Although these equations are nonlinear, one can use the same method for them as for linear equations: eliminate one variable and solve for the other.

3. This problem is worth **1.5 points**.
The purpose of this problem is to acquaint you with some typical solution of the Heat equation, which was derived in Section 9.1.

*Preamble*: The Heat equation has the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \tag{HW9.1$a$}$$

where we have set the coefficient $a$ in Eq. (6) to unity. Here $u(x,t)$ is the temperature of a rod, which extends from $x=0$ to $x=1$. The time is assumed to increase from the initial moment $t=0$. Thus, one is looking for a solution $u(x,t)$ of Eq. (HW9.1a) for

$$0 \le x \le 1 \qquad \text{and} \qquad t \ge 0. \tag{HW9.1$b$}$$

To find such a solution, one needs to supplement Eq. (HW9.1a) with an initial condition and boundary conditions. An example of such conditions is:

$$\text{initial}: \qquad u(x,0) = \cos(\pi x)\,; \tag{HW9.1$c$}$$

$$\text{boundary}: \qquad \frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(1,t)}{\partial x} = 0\,. \tag{HW9.1$d$}$$

It is shown in graduate-level courses that the boundary conditions (HW9.1d) correspond to thermo-insulated boundaries of the rod. That is, the thermal energy cannot leave the rod; it can only get redistributed along the rod as time goes by. This redistribution is govered by the Heat equation (HW9.1a) and the boundary conditions (HW9.1d).

(a) Verify that the function

$$u_m(x,t) = \exp[-(\pi m)^2 t]\,\cos(\pi m x), \tag{HW9.2}$$

where $m$ is an integer, satisfies the Heat equation as well as the boundary conditions stated above. Do the calculations by hand, not with Mathematica.
Also, find $u_m(x,0)$ for any $m$.

(b) Equation (HW9.2) by itself does not yet determine the temperature of the rod because the initial condition has not yet been specified. Let us specify it as

$$u(x,0) = \cos^2(\pi x). \tag{HW9.3}$$

Then the solution of the *initial-boundary value problem* consisting of Eqs. (HW9.1a,1d,3) is found using the *linear superposition principle* stated in Lecture 6: A linear combination of two (or more) solutions of a linear equation is also a solution of the same equation. (It applies here because the Heat equation is linear.) Your task in this part is to implement this principle for finding the solution of problem (HW9.1a,1d,3). Follow the steps listed below.

*Step 1*: Write the r.h.s. of (HW9.3) as a linear superposition of some $u_m(x,0)$, which you found in part (a). Your answer should look like

$$u(x,0) = c_{m1}u_{m1}(x,0) + c_{m2}u_{m2}(x,0) \tag{HW9.4}$$

for some $m_{1,2}$ and $c_{m1,m2}$.
*Hint*: Use a triginometric identity stated in Lectures 7 and 8.

*Step 2*: Write the solution of problem (HW9.1a,1d,3) as

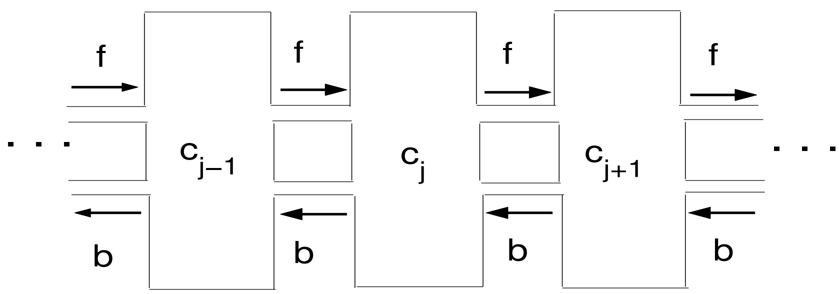
$$u(x,t) = c_{m1}u_{m1}(x,t) + c_{m2}u_{m2}(x,t) \tag{HW9.5}$$

with $m_{1,2}$ and $c_{m1,m2}$ that you have determined, and with $u_m(x,t)$ given by (HW9.2).
Food for thought: Notice how (HW9.5) is similar to (HW7.4) and to Eq. (5.9) in Lecture 5. They are express that good old principle of linear superposition, which holds for linear equations.

*Step 3*: Analytically (i.e., *not* with Mathematica), find the limit of your solution (HW9.5) as $t \to \infty$.
Describe briefly if this agrees with your intuition of how the temperature of a thermo-insulated rod[1] should behave.

*Step 4*: *In the same figure*, plot your solution (HW9.5) for $t = 0.05,\ 0.1,\ 0.2$ in the interval given in (HW9.1b). (When $t$ is fixed, $u(x,t)$ becomes a function of $x$ only, and all you are asked to do here is to plot that function.) Describe the observed behavior of this solution of the Heat equation as $t$ increases. State whether it agrees with your answer in the previous step.

4. This problem is worth **1.5 points**.
Consider the porous medium model mentioned at the end of Sec. 9.2:




Each element $\Delta x$ is modeled as a tank connected to its "neighbors" by two pipes, as schematically shown above. The rates of the flows to the right and to the left are maintained at values $f$ and $b$, respectively. Your derivation should follow that of Eq. (17), where you may assume $A = 1$ for simplicity.

(a) Derive — with an explanation — a difference equation with respect to $j$ for the concentration in tank $j$. This should be a generalization of Eq. (14) of the Notes.

(b) In the above model, relate the coefficients $f$ and $b$ to each other and to $\Delta x$ in such a way that in the limit $\Delta x \to 0$, you would obtain an equation of the form

$$\frac{\partial c}{\partial t} = r\frac{\partial c}{\partial x}\,, \tag{HW9.6$a$}$$

where $x$ is the continuous counterpart of index $j$. How is $r$ related to $f$, $b$, and $\Delta x$?

(c) Now, in the same model that you derived in part (a), relate the coefficients $f$ and $b$ to each other and to $\Delta x$ in another way, so that in the limit $\Delta x \to 0$, you would obtain an equation of the form

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}\,. \tag{HW9.6$b$}$$

How is $D$ related to $f$, $b$, and $\Delta x$?

(d) Finally, in the model of part (a), relate $f$ and $b$ to each other and to $\Delta x$ in such a way that in the limit $\Delta x \to 0$, you would obtain an equation of the form

$$\frac{\partial c}{\partial t} = r\frac{\partial c}{\partial x} + D\frac{\partial^2 c}{\partial x^2}\,. \tag{HW9.6$c$}$$

[1] Recall the paragraph after Eq. (HW9.1d).

In a slight variation from parts (b) and (c), express $f$ and $b$ in terms of $r$, $D$, and $\Delta x$ (not vice versa, as you did in parts (b) and (c)).
*Hint 1*: For example, you will find that $f = f_1(\Delta x)^{p_1} + f_2(\Delta x)^{p_2}$ (and similarly for $b$), where $f_{1,2}$ are constants independent of $\Delta x$ and $p_{1,2}$ are some integers.
*Hint 2*: Of course, your results for parts (b) and (c) should provide guidance for obtaining the result of (d).

5. (worh **0.5 point**)
In a simplified form, an income annuity works like this. One deposits a certain amount of money, say $K_0$ dollars, to a financial institution. (In other words, one buys an income annuity policy for $\$K_0$.) In return, the institution issues the depositor a fixed amount of money (say, monthly) and also compounds interest on the remaining amount.

Suppose a person deposits \$100,000 and expects to receive \$1,500 monthly for 15 years, at the end of which there should be exactly \$1,000 left on his/her account. What constant annual interest rate should the financial institution guarantee to fulfill this agreement?

*Suggestions*: Use the continuous approximation, as in the Section on mortgage payments, i.e., ignore the difference between continuous, monthly, or annual compounding of interest. Now, the resulting equation for the rate $r$ will be transcendental. You can solve an equation $f(x) = g(x)$ either graphically, by plotting $(f(x) - g(x))$ and finding where the graph intersects zero, or by Mathematica's command `NSolve`.

Bonus (worh **1 point**; as always with a Bonus problem, credit will be given only to mostly correct solutions)

This problem explores the continuous limit of the so-called Leslie model, summarized below.

Consider an age-structured population where all females of child-bearing age are divided into age groups, so that group $i$ with $i \geq 0$ contains females of age, say, from $13 + i\Delta t$ (years) to $13 + (i+1)\Delta t$ (years). (Group 0 then contains all females who are 0–13 years old.) At a given time, let the number of females in group $i$ be $x_i$. Suppose the birth rate of females in group $i$ ($i \geq 1$) is $b_i$, i.e., each female in group $i$ on average gives birth to $b_i$ females. Suppose also that the survival rate of females in group $i$ is $s_i$, i.e., $s_i \cdot 100\%$ of females who enter group $i$ reach the next age group. Finally, let the last child-bearing age group be for $i = m$. Then the evolution of the number $x_i$ of females in this model satisfies a linear system

$$\begin{pmatrix} x_0(t+\Delta t) \\ x_1(t+\Delta t) \\ \cdot \\ \cdot \\ \cdot \\ x_m(t+\Delta t) \end{pmatrix} = \begin{pmatrix} 0 & b_1 & \cdot & \cdot & b_{m-1} & b_m \\ s_0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & s_1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & s_{m-1} & 0 \end{pmatrix} \begin{pmatrix} x_0(t) \\ x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_m(t) \end{pmatrix}. \tag{HW9.7}$$

This model is named after Patrick Leslie, who was one of the first scientists to study, in the 1940s, the age-structured population dynamics using matrix theory.

Consider the case where $\Delta t \to 0$. Show that the continuous limit of (HW9.7) is given by an equation

$$\frac{\partial x}{\partial t} + \frac{\partial x}{\partial a} = \alpha(a)x \tag{HW9.8$a$}$$

*and* a boundary condition

$$x(0,t) = \int_0^A \beta(a)\, x(a,t)\, da\,, \tag{HW9.8$b$}$$

where $a$ is the continuous counterpart of index $i$, and $A = m\Delta t$. How are $\alpha(a)$ and $\beta(a)$ related to $s_i$, $b_i$, and $\Delta t$?