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VIII. On a New Type of Dynamical Stability.

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1. A system in a position of equilibrium, and capable of oscillation about that position, may be acted on by periodic force in such a way that no oscillation is generated. If, for example, one end of a vertical stretched string is moved to and fro in the direction of the length, this imposed motion has no tendency to produce lateral vibration. In certain cases, however, it has considerable effect in intensifying an already existing oscillation; in particular, if the imposed frequency is double that of the lateral motion, a very marked swing is magnified from the slightest initial disturbance.

Another example of a similar kind is afforded by a pendulum, the point of suspension of which is subject to a vertical periodic motion: if the frequency is double that of the pendulum, any small swing is gradually magnified by cumulative action.

Various instances of the double frequency effect forced themselves upon the attention of observers, and it appears to have been assumed—possibly from the simplicity of the phenomenon—that it is only in the case of double frequency that this type of disturbance has appreciable influence. Mathematical investigation has shewn, however, that the effect is cumulative in the whole series of cases when the disturbance frequency is approximately $2/r$ of the natural frequency of the system, r being any

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integer.* The intensity of the magnifying effect falls off rapidly when r is taken larger, and as our terrestrial systems are always subject to friction, it is difficult to exhibit many cases experimentally. With the pendulum the intensifying influence may be observed for several values of r without special nicety of adjustment.

Our present object is to establish another very remarkable property of the non-generating type of periodic disturbance.

2. In the preceding pendulum experiments the resulting motion is due to the combined action of the imposed force and gravity. Let us enquire as to the effect of the imposed force acting alone.

To examine this question experimentally, a rod is pivoted vertically so that it is free to rotate in a horizontal plane: when the pivot is driven horizontally in a simple vibration along the length of the rod the relative equilibrium is not disturbed. If, now, the rod is displaced through a small angle, it is observed to swing about the line of the imposed motion in a period large compared with that of the pivot.

All the properties of the motion may be deduced from the differential equation determining it, but here we seek an approximate treatment of the problem, based on general mechanical considerations, in order to obtain a notion of what happens more vividly than is possible from the exact solution. For this purpose we assume that a, the amplitude of the pivot motion, is small, and also that the speed of the pivot, P, is constant, and equal to V, say, throughout the path; *i.e.*, we assume that the body is acted on by suitable impulses applied at P at the ends of its path, being free from action in all intermediate

* "On a class of forced oscillations," *Quart. Journ. Math.*, No. 148, 1906.

positions. We shall first obtain the magnitudes and directions of the impulses necessary to impose the motion.

Let P be the instantaneous centre of rotation of a body of mass M, and with mass centre C. If an impulse, I, acts at P in a direction making a small angle, ϕ, with PC, it gives the mass centre a velocity I/M, and at the same time produces an angular velocity $Ih\phi/Mk^2$, where $h = PC$ and k is the radius of gyration about C.

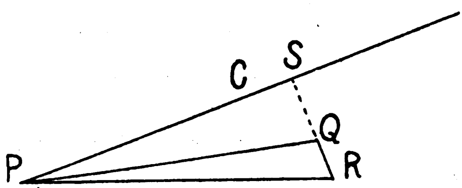

Fig. 1.

The velocity of P after the impulse is thus made up of the components $\overrightarrow{PQ} = I/M$ along the line of the impulse, and $\overrightarrow{QR} = (Ih^2/Mk^2)\phi$ perpendicular to PC: the resultant is therefore $\overrightarrow{PR}$. In our problem this velocity is constant (numerically), $= V$. PQR being approximately a right angle, PQ differs from PR by a small quantity of the second order: hence the impulse $I = M.PQ = MV$ so far as quantities of the first order are concerned. Also if RPC is denoted by θ

$$\frac{\theta}{\phi} = \frac{SR}{SQ} = \frac{h^2 + k^2}{k^2}.$$

Thus to impress the required velocity V on P in a direction at an angle θ to the rod an impulse MV must act at an angle $\frac{k^2}{h^2 + k^2}\theta$ to the rod. An impulse of double magnitude will then reverse the motion of P from V to $-V$: and the imposed vibratory motion of P with constant speed V in the path AB is produced by impulses of magnitude $2MV$ applied in the direction dividing the angle between the rod and AB in the constant ratio k^2/h^2. It will be convenient to regard the action at either end of the path as made up of two consecutive impulses, MV, one bringing P to rest, and the second giving it the velocity towards the other end. The angular velocity at the instant between the two impulses is then equal to the mean angular velocity during the cycle of which the instant is the mid point.

Consider the motion from the instant when P has been brought to rest at A by the action of the first half-impulse; let the inclination of PC to AB be θ and the angular velocity ω. The moment of the second half-impulse about C is

$$MV\frac{k^2h}{h^2+k^2}\theta$$

and the resultant change in the angular velocity

$$Vh\theta/(h^2+k^2).$$

If τ is the time of motion to B the inclination of the rod at B is therefore

$$\theta+\{\omega+Vh\theta/(h^2+k^2)\}\tau,$$

and the moment of the impulse at B

$$-2MV\frac{k^2h}{h^2+k^2}[\theta+\{\omega+Vh\theta/(h^2+k^2)\}\tau].$$

The angular velocity after this impulse is

$$\omega-Vh\theta/(h^2+k^2)$$

correct to small quantities of the first order, and the inclination on reaching A again is $\theta+2\omega\tau$. Hence the moment of the first half-impulse bringing P to rest at A is

$$MV\frac{k^2h}{h^2+k}(\theta+2\omega\tau).$$

By summation the total moment applied during the motion of P from rest to rest at A is

$$-2MV\frac{k^2h^2}{(h^2+k^2)^2}\theta\tau.$$

Hence if ω' is the angular velocity at the end of this cycle,

$$\frac{\omega' - \omega}{2\tau} = -\left(\frac{Vh}{h^2 + k^2}\right)^2 \theta \, ;$$

i.e., the mean angular acceleration is directed towards the position of relative equilibrium, and is proportional to the angular displacement. The mean motion is therefore a simple oscillation

$$\theta = a \sin\left(\frac{Vh}{h^2 + k^2} t + \epsilon\right) \quad . \quad . \quad . \quad (1)$$

The actual motion is evidently of the nature shewn in the diagram (*Fig.* 2), in which the time is plotted hori-

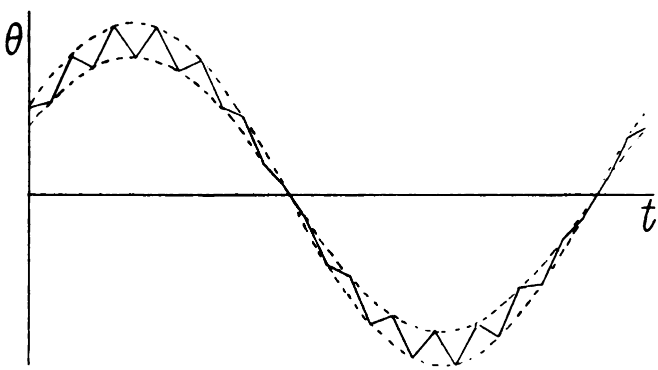

Fig. 2.

zontally and the angular displacement vertically: the two boundaries are sine curves and the successive vertices are equidistant in time.

The preceding synthetic investigation brings out the essential characteristics of the motion. The impulses are constant in magnitude, and the effect of any impulse in changing the angular velocity is proportional to the angular displacement; secondly, the angular displacement at *B* algebraically exceeds half the sum of the two at *A* on either side, by an amount proportional to the displacement; whence it follows that the impulsive moment in any cycle from rest to rest at A is always directed towards the central line of motion AB, and varies directly as the displacement—the conditions determining a mean motion of simple oscillation.

Such considerations tend to familiarise the motion, but the quantitative results may be obtained by a more analytical method, which will now be given for the sake of comparison.

Let θ be the inclination of PC to AB on starting from A after the impulse, and ω the angular velocity at the same time*; also let θ_1, ω_1 be the values of these quantities after the impulse at B, and θ_2, ω the values after the succeeding impulse at A.

On starting from A the velocity of the mass centre has components

$$(V - h\omega\sin\theta,\ h\omega\cos\theta)$$

along and perpendicular to AB. After the impulse at B the velocity is

$$(-V - h\omega_1\sin\theta_1,\ h\omega_1\cos\theta_1).$$

Hence if (X, Y) the impulse at B

$$X/M = -2V - h(\omega_1\sin\theta_1 - \omega\sin\theta)$$
$$Y/M = h(\omega_1\cos\theta_1 - \omega\cos\theta).$$

Again, considering the motion about the mass centre we have

$$Mk^2(\omega_1 - \omega) = (X\sin\theta_1 - Y\cos\theta_1)h$$
$$\therefore\quad k^2(\omega_1 - \omega) = -2Vh\sin\theta_1 - h^2(\omega_1 - \omega)\cos(\theta_1 - \theta),$$

and to the required degree of approximation

$$(h^2 + k^2)(\omega_1 - \omega) = -2Vh\theta_1 \quad . \quad . \quad . \quad \text{(i.)}$$

Similarly, considering the succeeding impact at A we have

$$(h^2 + k^2)(\omega_2 - \omega_1) = 2Vh\theta_2 \quad . \quad . \quad . \quad \text{(ii.)}$$
$$\therefore (h^2 + k^2)(\omega_2 - \omega) = 2Vh(\theta_2 - \theta_1) \quad . \quad . \quad \text{(iii.)}$$

* It must be remembered that ω here has not the same meaning as in the preceding.

Now, during the cycle, the mean angular velocity

$$=\frac{1}{2}(\omega+\omega_1)$$

$$=\omega-\frac{Vh}{h^2+k^2}\theta_1$$

$$=\omega\left(1-\frac{Vh}{h^2+k^2}\tau\right)-\frac{Vh}{h^2+k^2}\theta \quad . \quad . \quad . \quad \text{(iv.)}$$

Similarly, during the succeeding cycle, the mean angular velocity

$$=\omega_2\left(1-\frac{Vh}{h^2+k^2}\tau\right)-\frac{Vh}{h^2+k^2}\theta_2 \quad . \quad . \quad . \quad \text{(v.)}$$

Therefore, from (iv.) and (v.), the change in mean angular velocity

$$=(\omega_2-\omega)\left(1-\frac{Vh}{h^2+k^2}\tau\right)-\frac{Vh}{h^2+k^2}(\theta_2-\theta)$$

∴ from (iii.)

$$=\frac{Vh}{h^2+k^2}(\theta_2-2\theta_1+\theta)-2\left(\frac{Vh}{h^2+k^2}\right)^2(\theta_2-\theta_1)\tau$$

$$=\frac{Vh}{h^2+k^2}(\omega_1-\omega)\tau-2\left(\frac{Vh}{h^2+k^2}\right)^2(\theta_2-\theta_1)\tau$$

∴ from (i.)

$$=-2\left(\frac{Vh}{h^2+k^2}\right)^2(\theta_1-\overline{\theta_2-\theta_1})\tau$$

i.e. $\dfrac{\text{change in mean angular velocity in cycle } 2\tau}{2\tau}=-\left(\dfrac{Vh}{h^2+k^2}\right)^2\theta$

when small quantities of the second order are neglected.

Thus the mean angular acceleration is directed towards the position of equilibrium, and is equal to $\left(\frac{Vh}{h^2+k^2}\right)^2\times$ the angular displacement—the result obtained previously.

It is of interest to compare this case of constant pivot speed with the motion when the pivot has a simple vibration of small amplitude a, and frequency n per 2π units of time. The equation of motion is then

$$\ddot{\theta} + \frac{h}{h^2 + k^2} an^2 \cos nt \, \theta = 0$$

and the solution* when $ah/(h^2+k^2)$ is small, is approximately

$$\theta = \alpha \sin\left(\frac{1}{\sqrt{2}} \frac{anh}{h^2 + k^2} t + \epsilon \right) \quad . \quad . \quad . \quad (2)$$

It is evident that (1) and (2) are of the same type, an in (2) being the maximum velocity of P in its path.

We have thus proved that when the amplitude of the pivot motion is small, the body swings in a simple vibration of frequency proportional to the frequency and to the amplitude of the applied motion.

3. Now consider a body free to rotate about a horizontal pivot, and set in the position of unstable equilibrium: what will be the effect of a vertical oscillation of the pivot on the stability of the equilibrium?

In the position inclined to the vertical at an angle θ, the mean angular acceleration due to the imposed motion is $\frac{1}{2}\left(\frac{anh}{h^2 + k^2}\right)^2 \theta$ inwards, from (2); while the outward acceleration due to gravity is $\frac{gh}{h^2 + k^2}\theta$. The resultant is therefore

$$-\left\{ \frac{a^2 n^2 h}{2(h^2 + k^2)} - g \right\} \frac{h}{h^2 + k^2} \theta$$

and the acceleration is always towards the vertical if

$$(an)^2 > 2g(h^2 + k^2)/h.$$

Thus the inverted pendulum is rendered stable by a small simple vertical oscillation of the pivot of maximum velocity greater than

$$\sqrt{2g(h^2 + k^2)/h}.$$

* The investigation is given in a paper "On Induced Stability," *Phil. Mag.*, February, 1908.

When this condition is satisfied, the motion about the vertical is simple vibratory of frequency

$$\mu = \sqrt{\left\{\frac{a^2n^2h}{2(h^2+k^2)} - g\right\}\frac{h}{h^2+k^2}} \quad . \quad . \quad (3)$$

per 2π units of time.

To illustrate these results experimentally, a uniform rod of length 39·6 cm. was pivoted at one end, and the pivot was moved in an approximately simple vibration of amplitude 3·85 cm. With an applied motion of frequency 11·2 per sec., the period of the small oscillations about the vertical was found to be 1·64 sec. The above formula, (3), gives 1·58 sec. The 4% difference may be attributed partly to the effect of friction in lengthening the period, and partly to error in the determination of the pivot frequency.

If the imposed motion is slightly inclined to the vertical, it is observed that the pendulum makes small oscillations about a position much more markedly oblique in the same direction. This effect can be explained very simply.

Let the inclination of the applied motion be β, and that of the mean position of the pendulum γ. The accelerations due to the applied motion and gravity in this position must be equal and opposite; *i.e.*,

$$\frac{1}{2}\left(\frac{anh}{h^2+k^2}\right)^2(\gamma - \beta) = \frac{gh}{h^2+k^2}\gamma,$$

and therefore

$$\gamma = \beta \Big/ \left\{1 - \frac{2g(h^2+k^2)}{a^2n^2h}\right\}$$

Thus γ is large compared with β when n is near the limit necessary for stability.*

Finally, it may be noted that the stability effect still

* For the determination of the amplitude of the forced oscillation about the mean position, reference may be made to § 1 of the paper on "Induced Stability" already quoted.

holds when the rod is supported by a double joint so that it has complete freedom of motion about the vertical.

4. The particular case of dynamical stability investigated above is an example of a general type. If any system fixed by one coordinate is statically unstable in a position of equilibrium, that position is rendered stable by the action of a periodic disturbance applied in such a way as to generate no motion about equilibrium. The maintenance of the stability here does not necessarily demand an impressed motion: in the case of the inverted pendulum, for example, a periodic variation in gravity would have the same effect as the vertical oscillation of the pivot.

Some types of steady motion are also rendered stable by a non-generating periodic disturbance*; thus a top rotating at a speed too low for stability is maintained about the steady state by a vertical oscillation of the point of support.

It is possible that this method of ensuring the stability of a steady motion may be of service in special cases where the more usual devices are not applicable. In the problem of mechanical flight the great difficulty lies in obtaining longitudinal stability at slow speeds; if an aeroplane system is started in steady motion, a small disturbance results in a pitching oscillation, which continues with increasing violence until finally the glider is overturned. The mathematical investigation of the effect of the non-generating periodic disturbance which is illustrated in this paper, was undertaken with the view of its possible application to a mechanism whereby stability in gliding would be automatically ensured. In such a case the motion is of a more complex character, involving the interaction of several co-ordinates: it is hoped to give a general examination of the problem later.

* *loc. cit.*, § 2 and 3.