NOTES

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What Goes Up Must Come Down, Eventually

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1. The motion of a particle falling under the action of a constant gravitational force and a retarding force of friction proportional to velocity is often presented as an example in courses in calculus, elementary physics, and elementary differential equations. The closely related problem of determining the motion of a particle propelled upwards from the surface of the earth is a little more complicated technically. Intuition tells us that the particle will rise to a maximum height and then fall back to earth. Intuition also suggests that since friction acts downward when the particle is rising and tends to shorten the rising time while friction acts upwards when the particle is falling and tends to lengthen the falling time, the particle will take longer to fall than to rise. It is non-trivial to show that this is indeed true, but the argument is not beyond the understanding of an elementary class; see [**1**, p. 39], [**6**, p. 526]. The approach is to find the time t_m taken for the particle to reach its maximum height $y(t_m)$, and then to show that $y(2t_m) > 0$. This means that the rising time is t_m while the falling time is more than t_m.

The assumption that the force of friction is proportional to velocity is convenient, as it produces a linear differential equation as a model. However, it is quite controversial. In many situations, experiments suggest that it is a plausible assumption, at least as an approximation [**4**]. However, there are other situations in which it is far from a good approximation. For example, it appears that the drag in air on a skydiver or parachutist is approximated reasonably well by a constant multiple of the square of the velocity [**5**]. Also, it has been suggested that the drag on a golf ball is approximated well by a constant multiple of $v^{1.3}$. However, the motion of a golf ball is more complicated because of its backspin [**2**]. More generally, the force of friction depends on many factors, including the density and viscosity of the fluid in which the particle is travelling and the shape of the particle. The Reynolds number is a dimensionless quantity that is important in determining the drag force. For very small Reynolds numbers the drag force is approximated well by a linear function of velocity, while for somewhat larger Reynolds numbers the drag force is approximated better by a quadratic function [**5**].

It is natural to ask whether a particle propelled upwards takes longer to fall to earth from its maximum height than it takes to rise to this maximum height for frictional forces that are nonlinear functions of velocity. Since linear and quadratic retarding forces are at best approximations, we would like to answer the question for a general force function. The purpose of this Note is to establish that the falling time is greater than the rising time in general.

2. Let $y(t)$ be the height of a particle above the surface of the earth at time t, and let $v(t) = y'(t)$ be the velocity at time t. Then upward motion corresponds to positive velocity. We assume that the particle is projected vertically upward from the surface of the earth at time $t = 0$ with a positive initial velocity v_0. We assume that the only forces acting on the particle are a constant downward gravitational force $-mg$ and a retarding force of function $-f(v)$. The retarding nature of the force of friction is described by the requirement

$$vf(v) > 0 \quad (v \neq 0), \tag{1}$$

i.e., $f(v) > 0$ if $v > 0$ and $f(v) < 0$ if $v < 0$. From physical considerations it might also be reasonable to assume that $f(v)$ is monotone non-decreasing in v for $-\infty < v < \infty$, but we do not need to make any such assumption. We do, however, require that the function f is smooth enough to guarantee uniqueness of solutions of the initial value problem (2); continuous differentiability of f is ample for this purpose.

The motion may be described by the autonomous first order initial value problem

$$mv' = -mg - f(v), \quad v(0) = v_0. \tag{2}$$

Then the height $y(t)$ may be obtained by integration of the solution for v with $y(0) = 0$. For $t = 0$, $v(0) = v_0 > 0$ and $v'(0) = -g - f(v_0)/m < 0$. Thus v decreases initially, and since solutions of an autonomous first order differential equation are monotone, v decreases for all time. An equilibrium of (2) is a solution of

$$mg + f(v) = 0,$$

and since $f(v) > 0$ for $v > 0$, all equilibria are negative. If (2) has no equilibria, v continues to decrease and $\lim_{t\to\infty} v(t) = -\infty$. If (2) does have at least one equilibrium and if v^* is the largest equilibrium of (2), then $\lim_{t\to\infty} v(t) = v^*$, and v^* is the terminal velocity. In either case, v is positive on some interval $0 \leq t < t_m$ and v is negative for $t > t_m$. Thus the particle rises to a maximum height at t_m and then falls for $t > t_m$.

On the interval $0 < t \leq t_m$, separation of variables in (2) gives

$$\int_{v_0}^{v(t)} \frac{m\,dv}{mg + f(v)} = -t.$$

Since $v(t_m) = 0$, we have

$$t_m = \int_0^{v_0} \frac{m\,dv}{mg + f(v)}. \tag{3}$$

On $0 \leq t < t_m$, $v > 0$ and therefore $f(v) > 0$.

For $t = t_m$, we use $v(t_m) = 0$ as an initial condition, giving the initial value problem

$$mv' = -mg - f(v), \quad v(t_m) = 0. \tag{4}$$

For $t \geq t_m$, $v < 0$ and therefore $f(v) < 0$. Separation of variables in (4) gives

$$\int_0^{v(t)} \frac{m\,dv}{mg + f(v)} = -(t - t_m). \tag{5}$$

If we take $t = 2t_m$ in (5), we obtain

$$t_m = \int_{v(2t_m)}^{0} \frac{m\,dv}{mg + f(v)}. \tag{6}$$

Comparison of (3) and (6) gives

$$\int_0^{v_0} \frac{mdv}{mg + f(v)} = \int_{v(2t_m)}^0 \frac{mdv}{mg + f(v)}. \tag{7}$$

In the integral on the left side of (7), $v > 0$ and thus $f(v) > 0$, while in the integral on the right side of (7), $v < 0$ and thus $f(v) < 0$. Thus the integrand on the left side of (7) is less than the integrand on the right side of (7). In order to have equality in (7), we must have

$$v_0 = v(0) > -v(2t_m). \tag{8}$$

We may use the same argument for every initial point $(\tau, v(\tau))$ with $0 \le \tau \le t_m$ on the rising part of the solution curve $v(t)$. Our interpretation of (8) is that if $v(\tau)$ is the velocity at time τ, the rising time is $t_m - \tau$ and the velocity $(t_m - \tau)$ after t_m is $v(2t_m - \tau)$ and it satisfies

$$v(\tau) > -v(2t_m - \tau) \tag{9}$$

for every $\tau, 0 \le \tau \le t_m$. Integration of (9) using $y'(t) = v(t)$, $y(0) = 0$ gives

$$y(t_m) = \int_0^{t_m} v(\tau)d\tau > -\int_0^{t_m} v(2t_m - \tau)d\tau = y(t_m) - y(2t_m).$$

This implies $y(2t_m) > 0$ and thus establishes that the time to fall from maximum height is greater than the time to rise to maximum height for arbitrary retarding force of friction.

3. Introductory courses in differential equations (and introductions to differential equations in calculus courses) have always concentrated on techniques for explicit solution, usually beginning with the method of separation of variables. Recently, it has become common to include some geometric and qualitative ideas, certainly direction fields and possibly stability analysis of equilibria of first order autonomous equations. Our purpose in this Note is to obtain some qualitative information about the behaviour of solutions by beginning with an explicit solution method and using it in a qualitative way. The result that we obtain by this approach is much more general than the result obtained by explicit solution in the case of a linear force of friction, and the proof is actually more concise. However, the result is obtained by following the argument of the linear case; in order to discover the general result one must first work out the special case.

In [**3**], C. Groetsch analyzes the two-dimensional projectile motion problem of a projectile sent up from the surface of the earth in a non-vertical direction with a frictional force that is a function of the magnitude of the velocity vector. He is able to obtain some qualitative information about the nature of the trajectory. However, an estimate of the relation between the rising and falling appears to be a much more difficult question.

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Designing a Calculational Proof of Cantor's Theorem

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Cantor's Diagonalization The one purpose of this little Note is to show that formal arguments need not be lengthy at all; on the contrary, they are often the most compact rendering of the argument. Its other purpose is to show the strong heuristic guidance that is available to us when we design such calculational proofs in sufficiently small, explicit steps. We illustrate our approach on Georg Cantor's classic diagonalization argument [chosen because, at the time, it created a sensation].

Cantor's purpose was to show that any set S is *strictly* smaller than its powerset $\wp S$ (i.e., the set of all subsets of S). Because of the 1-1 correspondence between the elements of S and its singleton subsets, which are elements of $\wp S$, S is not larger than $\wp S$, and our proof can now be focussed on the "strictly", i.e., we have to show that there is no 1-1 correspondence between S and $\wp S$. We can confine ourselves to non-empty S.

1. PROOF FORMAT AND NOTATION.

Eventually we present our proof in a format, due to W.H.J. Feijen, in which consecutive proof stages are separated by a connective and a justification. Thus,

$$\begin{array}{ll} & p \\ \Rightarrow & \{J\} \\ & q \\ \equiv & \{M\} \\ & r \end{array}$$

would show a proof of $p \Rightarrow r$ in which J justifies the conclusion q from p, while M explains why q and r are equivalent. In our proof we use $\equiv$ and $\Leftarrow$, the latter connective being the converse of $\Rightarrow$, i.e., $(p \Rightarrow q) \equiv (q \Leftarrow p)$.

In writing quantified formulae, we use the angle brackets, $\langle\rangle$, to delineate the scope of the dummy, and the double colon, ::, to separate the dummy from the quantified term, as in $\langle\forall x :: p.x\rangle$. Function application, as in the preceding "$p.x$", is denoted explicitly by an infix dot.

Besides "substituting equals for equals", we use the Rule of Instantiation, viz., that for any expression y in the range of the dummy x

$$\langle\forall x :: p.x\rangle \Rightarrow p.y.$$