The Calculus of Rainbows

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Abstract

The geometry of reflection and refraction explains the apparent position of a rainbow relative to the sun, and calculus shows why light is concentrated in the rainbow. Exercises include the derivation of Snell's Law and the Law of Reflection, an explanation for the different colors in the rainbow, and an exploration of secondary and tertiary rainbows. This is intended as a one-hour lecture for honors calculus, with exercises which students can work through on their own.

Observation of Rainbows

What do you need to make a rainbow? Of course, you need at least two things, rain and sun. Closer examination reveals more details–but not more ingredients! Not all rainbows are alike. Suppose you see a rainbow at sunset (or sunrise).

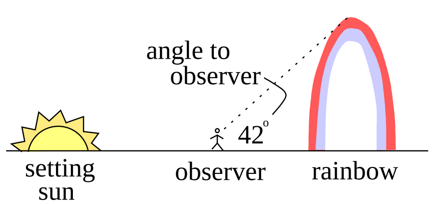

Fig. 1

A rainbow made by the setting sun appears to be a semicircle centered at the point on the horizon opposite the sun, with the angle of elevation from the observer to the top of the rainbow being about 42°. Earlier in the day, the picture is a little different.

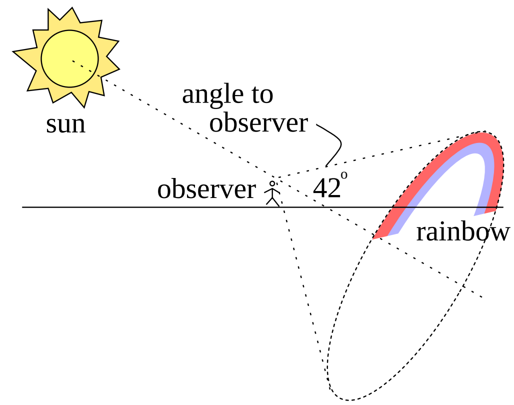

Fig. 2

The rainbow is at the base of an imaginary cone, whose

- tip is at the observer,
- axis is the line from the observer to the sun, and
- angle between side and axis is 42°.

You can create rainbows–all you need is a sunny day and a garden hose or spray bottle which can produce a fine mist. With a little experimentation, you should be able to see behavior similar to Figure 3.

Another phenomenon which you may have observed is the secondary rainbow. When the conditions for rainbows are especially good, there will be another, fainter rainbow above the primary rainbow. If you have a protractor handy, you will notice that the angle of elevation of the secondary rainbow is about 51°. When conditions are *extremely* good, might it be possible to see a still fainter tertiary rainbow? Or an infinite

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series of fainter rainbows?[1]

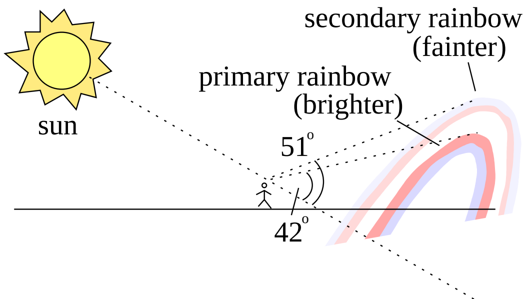


FIG. 3

What Happens in the Raincloud?

Let's focus more closely on the raincloud which produces the rainbow. Light comes from the sun and encounters drops of water. The direction of the light is altered by two phenomena we'll discuss in detail–*refraction* and *reflection*.

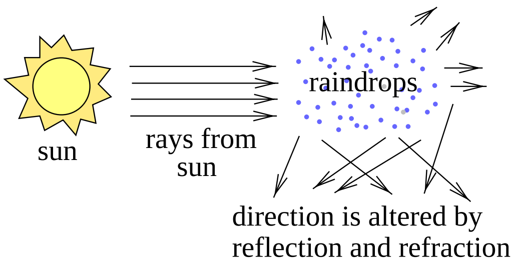
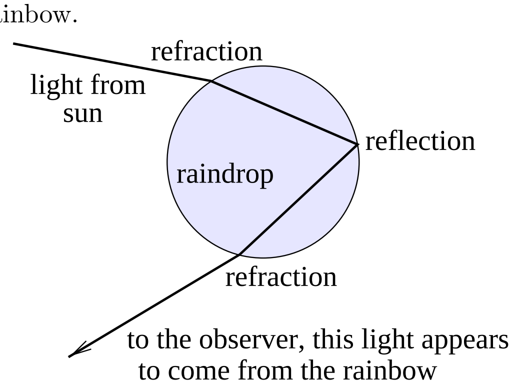


FIG. 4

In fact, the complex behavior which produces a rainbow can be explained by looking at a single raindrop. Here's a sketch which shows the path of light travelling through a raindrop to create a rainbow.



[1] There is a difference between mathematical possibility and physical possibility!

FIG. 5

In order to understand Figure 6, we'll have to know a little about the behavior of light as it travels from air to water, and back again.

Law of Refraction (Snell's Law)

Light travelling from source to observer takes the quickest path. Usually this is a straight line, but if the light travels through different media (say, both air and water) it will change direction slightly due to the fact that the speed of light varies in different media. For instance, the speed of light is slower in water than in air, which causes the path to bend at the point of entry.

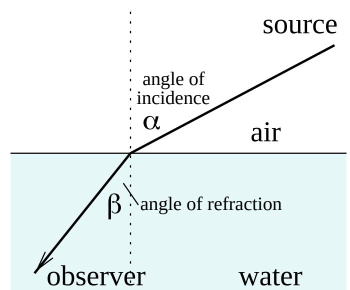


FIG. 6

Why? The most direct path from source to observer travels through too much water to be the quickest, but the path minimizing the amount of water the light travels through is too long to be the quickest. The actual quickest path is a compromise between these two.

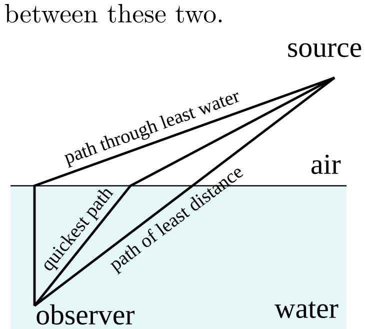


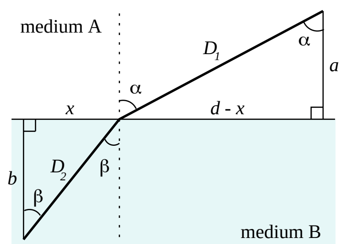
FIG. 7

In fact, we can compute the path exactly! The exercise below shows that the path of least time is found when

$$\sin(\alpha) = k \sin(\beta)$$

where α and β are the angles of incidence and refraction, respectively, and k is the ratio of the speed in the source medium to the speed in the observer's medium. Notice that if the positions of the source and observer are reversed (so that the source is in water and the observer is in air), the quickest path between the two is unchanged. So light travelling from water with incidence angle β will have angle of refraction α.

Exercise 1 Follow the steps below to verify that the quickest path from source to observer is given when $\sin(\alpha) = k\sin(\beta)$.



1. In the diagram above, a, b, and d are fixed, and the other variables are functions of x. Find D_1 and D_2 in terms of x.
2. Let v_1 be the speed of light in medium A and v_2 the speed in medium B, and show that the time taken to complete the path is
$$\frac{\sqrt{a^2+(d\quad x)^2}}{v_1}+\frac{\sqrt{b^2+x^2}}{v_2}.$$
3. Now use the derivative to show that the above quantity is minimized when
$$\frac{d\quad x}{D_1 v_1}=\frac{x}{D_2 v_2}$$
and conclude Snell's Law (let $k=\frac{v_1}{v_2}$).

Law of Reflection

Light which strikes off a mirror or other surface has the following behavior:

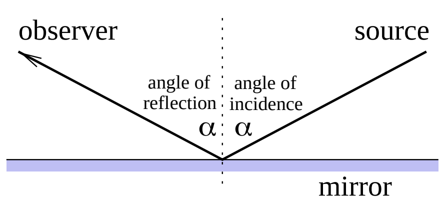


FIG. 8

In other words,

$$\text{angle of incidence} = \text{angle of reflection.}$$

It's interesting to note that refraction and reflection are two manifestations of a single law, called Fermat's Principle, which states that the light which reaches your eye is the light which has travelled along the quickest path from its source.

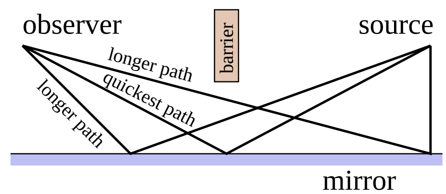


FIG. 9

Whether you look at your hand in a mirror or underwater, the rays you see are those which have taken the quickest path to your eye.[2]

The Turning Angle

Now that we can account mathematically for some of the behavior of light, let's revisit the raindrop model. To make things simpler we'll suppose that the raindrop is perfectly spherical, and that it makes sense to look at everything in two dimensions. Suppose light strikes the drop with incidence angle α, is reflected by the back wall of the drop, and then is refracted again on leaving the drop. By what angle is light turned by its encounter with the raindrop?

[2] Actually the behavior of light is more complicated than that. Light can travel along *all* possible paths from the source to the observer–but those rays travelling very close to the quickest path reinforce each other, while the others cancel each other out, so *all* the light appears to have taken this path. But that's another story...(*cf* Feynman, QED [1]).

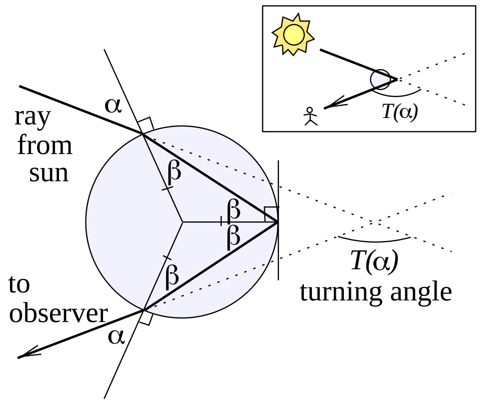


FIG. 10

Let α be the angle of incidence and β the angle of refraction, which is related to α by Snell's Law. As the ray travels to the far side of the drop, it moves along the base of an isoceles triangle whose sides are both radii of the drop. After it reflects off the inside wall of the drop, the ray travels back through the drop and completes another, *identical* isoceles triangle. As the picture shows, the incidence angle for the light leaving the drop will now be β. As we have already noted, light travelling from water into air with incidence angle β will have angle of refraction α.

Suppose $T(\alpha)$ is the turning angle, that is, the total angle by which the ray is turned by its encounter with the raindrop, measured clockwise from a straight path.

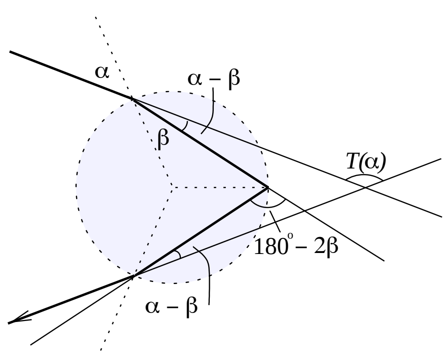


FIG. 11

Light entering the drop is turned by $\alpha - \beta$ (why?) as is light leaving the drop. The reflection causes a turn of $180° - 2\beta$. We can add up all the angles to get

$$\begin{aligned} T(\alpha) &= (\alpha - \beta) + (180° - 2\beta) + (\alpha - \beta) \\ &= 180° + 2\alpha - 4\beta. \end{aligned}$$

So light entering the drop at an angle α is turned by the angle $180° + 2\alpha - 4\beta$.

Concentration of light

We've seen that if light enters a raindrop at an angle α then (some of) it is turned by an angle $T(\alpha)$. But this doesn't explain why you only see the turning effect along a certain band which is 42° above the axis to the sun.

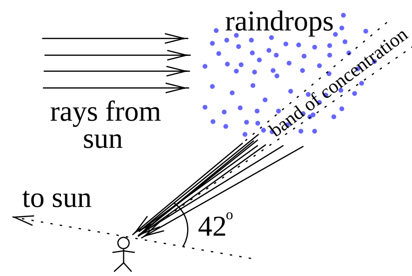


FIG. 12

What happens is that although in fact light is turned through various angles, more light (much more) reaches the observer from the marked band than from elsewhere.

A little calculation—and a little calculus—shows why. We're going to compute the rate of change in turning angle $T(\alpha)$ with respect to α. In other words, we're going to figure out $\frac{dT}{d\alpha}$. After we've done, and after we've figured out for which values of α the derivative $\frac{dT(\alpha)}{d\alpha}$ is zero, we'll consider what all this calculus has to do with the concentration of light at 42°.

Differentiating both sides of

$$T(\alpha) = 180° + 2\alpha - 4\beta$$

by α gives

$$\frac{dT}{d\alpha} = 2 - 4\frac{d\beta}{d\alpha}.$$

But what is $\frac{d\beta}{d\alpha}$? We can differentiate both sides of the refraction law

$$\sin(\alpha) = k\sin(\beta)$$

by α to get

$$\cos(\alpha) = k\cos(\beta)\frac{d\beta}{d\alpha},$$

and putting it all together,

$$\frac{dT}{d\alpha} = 2 - \frac{4\cos(\alpha)}{k\cos(\beta)}.$$

Remember that the derivative of a function evaluated at a point allows us to find the linear approximation to the function at that point. That is,

$$T(\alpha) \approx T(\alpha_0) + T'(\alpha_0)(\alpha - \alpha_0)$$

as long as $\alpha - \alpha_0$ is small. If we can find a point α_0 where $T' = 0$, then $T(\alpha) \approx T(\alpha_0)$ for any α close to α_0. This means precisely that every ray of light entering with incidence angle near the angle where $T' = 0$ is turned by approximately the same amount.

Now let's find α_0. Suppose

$$0 = \frac{dT}{d\alpha} = 2 - \frac{4\cos(\alpha_0)}{k\cos(\beta_0)}.$$

With a little algebraic manipulation,

$$k^2\cos^2(\beta_0) = 4\cos^2\alpha_0,$$

and hence

$$k^2(1 - \sin^2(\beta_0)) = 4(1 - \sin^2(\alpha_0)).$$

Now use the law of refraction to substitute for $\sin(\beta_0)$

$$k^2 - \sin^2(\alpha_0) = 4 - 4\sin^2(\alpha_0)$$

and solve for $\sin(\alpha)$

$$\sin^2(\alpha_0) = \frac{1}{3}(4 - k^2).$$

If $k = 1.33$, then $\alpha_0 = 59.4°$ and so $T(\alpha_0) = 138°$.

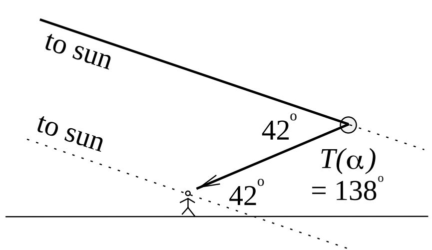


FIG. 13

Since the line from the raindrop to the sun and the line from the observer to the sun are approximately parallel, the angle of elevation from the line to the sun is $180° - 138° =$ (presto!) $42°$, our "magic number".

So all the rays for which $\alpha - \alpha_0$ is small will appear to come from the same point, precisely $42°$ from the axis to the sun. To demonstrate this point graphically, let's assume each value of α between $0°$ and $90°$ is equally likely, and find the distribution of $T(\alpha)$.

Choosing the values $\alpha = 0.0°, 0.5°, 1.0°, ..., 89.5°, 90.0°$ we obtain the following frequency distribution for $T(\alpha)$:

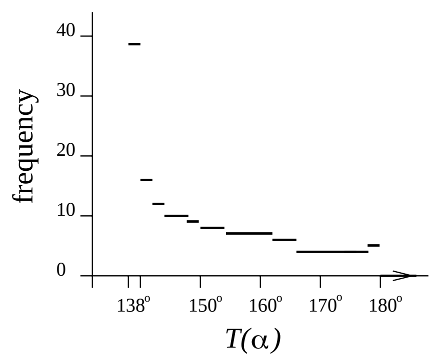


FIG. 14

The "spike" around $138°$ is what produces the band of light at $42°$.

Exercise 2 When calculating the distribution of $T(\alpha)$, we assumed that all values of α, $0° \leq \alpha \leq 90°$, are equally likely. Actually, assuming the light rays entering the drop are parallel and equally distributed, what is the actual distribution of α?

The Colors of the Rainbow

So far, we've explained why the rainbow appears as a band of light in the sky, but we haven't said anything about the colors of the rainbow. The color separation in a rainbow is due to the fact that the constant k in the Law of Refraction is slightly different for different colors of light. When white light is refracted, each of the component colors is turned by a different amount.

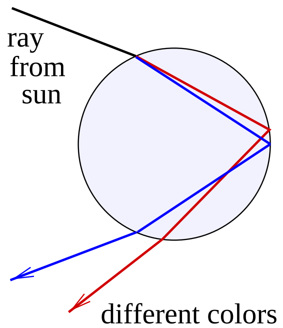


FIG. 15

Exercise 3 For red light, $k \approx 1.3318$, while for violet light, $k \approx 1.3435$. Should red appear above or below violet?

Secondary Rainbows

Why does the light sometimes reflect in the raindrop and at other times pass through it? The answer is that only *some* of the light exhibits the behavior we've describing–each time the light encounters the wall of the drop, some of it is reflected and some "escapes".

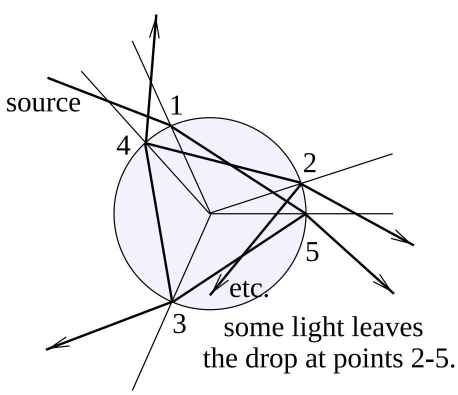


FIG. 16

The point is that enough light follows the path described above that we can see a rainbow.

The phenomenon of the secondary rainbow appears when light strikes below the "equator" of the raindrop and is reflected *twice* before leaving the drop.

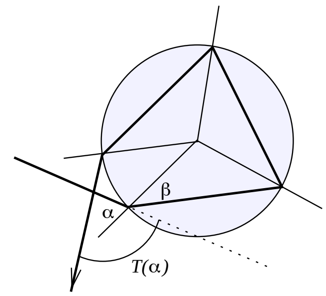


FIG. 17

It is fainter since light is less likely to reflect twice than to reflect once.

Exercise 4 Let's investigate the secondary rainbow.

1. Where should the secondary rainbow appear? Show that the turning angle in this case is given by

$$T(\alpha) = 360^\circ + 2\alpha \quad 6\beta$$

and follow the same steps as above to show that $\frac{dT}{d\alpha}$ is 0 when $T(\alpha)$ is approximately 129°.

2. Will red be on the top or bottom? Find the turning angle for red and violet light in the secondary rainbow.

3. How and where might a tertiary rainbow appear? Will red be on the top or bottom? Is it possible that fourth or fifth rainbows might appear?

References

[1] Richard P. Feynman. *QED : the strange theory of light and matter*. Princeton University Press, Princeton, New Jersey, 1985.