The Track of a Bicycle Back Tire

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A rider on a bicycle goes down a road, steering a path with the front tire, or perhaps just weaving back and forth. If the tires pass through a puddle, we can see from the tracks that the back tire follows a path similar to that of the front tire. Suppose we know the path of the front tire precisely; what is the path of the back tire? A related question is: If the front tire travels some distance, how far does the back tire travel? It seems obvious from experience that the back tire travels a shorter path than the front tire. Bicycle folklore says that after a long trip the back tire will show about 10% less wear than the front tire. Is it possible to verify the folklore?

In this article we derive and solve differential equations for the path of the back tire, given the path of the front tire. For a parametric form of the front-tire path the differential equations for the path of the back tire are a pair of coupled nonlinear differential equations.

These equations for the back-tire path are a simple example of vector Riccati equations. In a few idealized cases, we can solve them directly, with geometrical arguments or by "guess-and-check." More realistic cases require more sophisticated methods. We use both regular perturbation (matching terms of the solution's Taylor expansion in terms of a small parameter) and iteration (successive approximation) techniques. We'll then derive some quantitative rules for the distance the back tire travels compared to the front tire.

The formulation and solution of the differential equations for the back-tire path given the front-tire path is an example of a *forward* or *direct problem*. In contrast, an *inverse problem* would be: Given the paths of both the front tire and the back tire, determine which was the path of the front tire. This inverse problem is explored in the well-known article, "Which way did the bicycle go?" [**9**], which leads to an elementary calculus problem relating the tangent lines of the two paths. We use this same relation of the tangent lines in formulating the differential equations for the direct problem. The solution of the direct problem will also give another way to characterize the front- and back-tire paths in terms of magnitude of the oscillations. This provides another elementary way to solve the inverse problem in special cases when the paths are sinusoidal.

Investigating the dynamics of bicycles has been a favorite topic in physics for a long time, and there are many references in the physics education journals [**1, 2, 4, 5, 3, 6, 7**], particularly with reference to the stability of a bicycle. There are fewer references to bicycles in the mathematics literature [**9, 10**], generally dealing with the paths of the tires. This article provides an elementary application of coupled nonlinear differential equations to a familiar situation. The application and methods are suitable for classes on methods of applied mathematics and differential equations. A *Maple* worksheet showing the computations and figures is available at the MAGAZINE website: http://www.maa.org/pubs/mathmag.html.

Model equations

Assume that the path of the contact point of the front tire of a bicycle is given parametrically as a function of time as $(x_f(t), y_f(t))$. The contact point is where the tire touches the road. The problem is to determine the path of the contact point of the rear tire, represented parametrically by $(x_b(t), y_b(t))$. Let a be the wheel-base of the bike, that is, the distance between the front and back contact points. As the handlebars are turned, the front tire more or less swivels on the contact point below. Actually, depending on the design of the bicycle, the front contact point moves slightly relative to the back, but we will treat a as a constant. Analyzing how this assumption affects the results would be a good exercise in modeling. The dimensions of a, $x_b(t)$, $y_b(t)$, $x_f(t)$ and $y_f(t)$ are in the units of distance.

We draw on a primary physical fact (the main ingredient in [**9**]): the tangent vector to the path of the rear tire always points to the contact point on the path of the front tire. Since the wheel-base is constant, we may express this fact as a pair of differential equations for the path of the rear-tire contact point:

$$x_b(t) + a\frac{x_b'(t)}{\sqrt{(x_b')^2 + (y_b')^2}} = x_f(t)$$

$$y_b(t) + a\frac{y_b'(t)}{\sqrt{(x_b')^2 + (y_b')^2}} = y_f(t).$$

This pair of equations can be rearranged as

$$\frac{x_b'(t)}{\sqrt{(x_b')^2 + (y_b')^2}} = \frac{x_f(t) - x_b(t)}{a} \tag{1a}$$

$$\frac{y_b'(t)}{\sqrt{(x_b')^2 + (y_b')^2}} = \frac{y_f(t) - y_b(t)}{a}. \tag{1b}$$

This says that the unit velocity vector for the back tire points in the direction (unit vector!) of the wheel-base of the bicycle. Taking a closer look, equation (1) is a relation between two unit vectors. There is no way to determine the magnitude of the velocity vector $(x_b'(t), y_b'(t))^T$. With this in view, call the speed of the back tire point $v(t) = \sqrt{(x_b')^2 + (y_b')^2}$. Rewriting (1) gives

$$x_b'(t) = v(t)\frac{x_f(t) - x_b(t)}{a} \tag{2a}$$

$$y_b'(t) = v(t)\frac{y_f(t) - y_b(t)}{a}. \tag{2b}$$

If we can express the speed of the back-tire point in terms of the known front-tire speed, then we will be able to express the differential equation for the back-tire path in terms of $(x_b(t), y_b(t))$ and known quantities.

Think of the back-tire velocity as though the velocity vector of the front tire is dragging the back tire along. The velocity of the back tire will be the component of the front-tire velocity in the direction of the back-tire motion. The back-tire speed is the magnitude of this projection, found using the dot product:

$$v(t) = \begin{pmatrix} x_f'(t) \\ y_f'(t) \end{pmatrix} \cdot \begin{pmatrix} \frac{x_f(t)-x_b(t)}{a} \\ \frac{y_f(t)-y_b(t)}{a} \end{pmatrix} \tag{3}$$

With this, the right sides of equations (2) are expressed in terms of the given quantities $(x_f(t), y_f(t))$ and $(x_f'(t), y_f'(t))$ and the unknown track $(x_b(t), y_b(t))$. This puts our equations (2) into a nice form: derivatives of the unknown quantities are expressed in terms of the unknown quantities and various known quantities.

Expression as a Riccati equation

We rewrite the equations for the bicycle back tire in an alternative form, relating them to the standard theory for matrix Riccati equations. Although we will not pursue solving the equations in this form, the matrix form suggests some other approaches to analyzing the equations theoretically and solving them efficiently with numerical methods.

This application of matrix Riccati equations to bicycle tire tracks is a very elementary and easily derived example; other less elementary applications of Riccati equations arise in transmission line theory, random noise theory, variational equations, and control theory [**8**].

Insert expression (3) into (2) and collect terms

$$x_b'(t) = \frac{1}{a^2}\left[x_f'(t)\left\{x_f(t)-x_b(t)\right\}^2 + y_f'(t)\left\{x_f(t)-x_b(t)\right\}\left\{y_f(t)-y_b(t)\right\}\right] \tag{4a}$$

$$y_b'(t) = \frac{1}{a^2}\left[x_f'(t)\left\{x_f(t)-x_b(t)\right\}\left\{y_f(t)-y_b(t)\right\} + y_f'(t)\left\{y_f(t)-y_b(t)\right\}^2\right]. \tag{4b}$$

Now the quadratic form characteristic of Riccati equations is explicit, but cumbersome. For a more elegant presentation, we write the equations in terms of the quantities $w_1(t) = x_b(t) - x_f(t)$, and $w_2(t) = y_b(t) - y_f(t))$. A little work leads to:

$$w_1'(t) = \frac{1}{a^2}\left[x_f'(t)w_1^2(t) + y_f'(t)w_1(t)w_2(t)\right] - x_f'(t) \tag{5a}$$

$$w_2'(t) = \frac{1}{a^2}\left[x_f'(t)w_1(t)w_2(t) + y_f'(t)w_2^2(t)\right] - y_f'(t). \tag{5b}$$

Let $W(t) = [w_1(t), w_2(t)]^T$ be a 2×1 matrix (or vector). Then equations (5) can be expressed as

$$W'(t) - \frac{1}{a^2}\left[W(t)\,F^T(t)\,W(t)\right] + F(t) = \mathbf{0},$$

where $F(t) = [x_f'(t), y_f'(t)]^T$. This is a special case of the general Riccati matrix differential equation treated by Reid [**8**, equation (2.1), page 11], where linear terms are allowed in addition to the quadratic term in our equation. This demonstrates the remarkable likeness between the equations for such simple things as bicycle paths and complicated things like random noise theory. Matrix Riccati equations can arise even from familiar physics.

Some simple paths for the front tire

We check our modeling by seeing that the differential equations predict the right behavior in cases where we already know what happens.

Simple case I: A straight path, no turns, constant front-tire speed Suppose the path of the front tire is a straight line, say along the x-axis. Our experience tells us that if the front-tire path is a straight line, then the back tire should follow behind on the same straight line. We leave it to the reader to verify this directly from the differential equations, which are easy to solve in this case. This is recommended as a way to get to know the equations better. Set

$$x_f(t) = v_f\, t,\ \ y_f(t) = 0.$$

After writing down the differential equations for $x_b(t)$ and $y_b(t)$, it is easy to check that the solutions are

$$x_b(t) = -a + v_f\, t,\ y_b(t) = 0.$$

Thus, the back tire follows a straight line, lagging by only the wheel-base of the bicycle.

Simple case II: A large circular path Take the path of the front tire to be a large circle. From our experience riding a bike around a circle, the back-tire contact point trails behind and inside the circular front-tire path. In steady state, the angle between the circle and the bike body (wheel-base) should be constant, as shown by the symmetry of the system. Since the bike is of constant length and makes a constant angle, the back-tire contact point follows a concentric circular path. The situation is depicted in FIGURE 1. The question now becomes: "What is the radius of the inner circle of the back tire?".

Set some notation: Let c be the radius of the large circle traced by the front-tire contact point. Let a be the wheel-base of the bike. Let b be the (unknown) radius of the inner circle traced by the back tire. Remember that the velocity vector of the back tire contact point will point at the contact point of the front tire. But the velocity vector of the back-tire contact point is perpendicular to the radius vector of the back-tire contact point. As seen in the diagram, a right triangle is formed by the back-tire

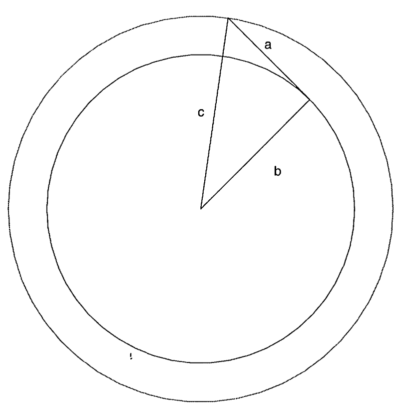

Figure 1 The tracks of the front and back tires, the radii, and the bicycle wheel-base when the front tire follows a circular path

contact point radius vector (of length b), the bike wheel-base vector (of length a), and the front-tire contact point radius vector (of length c). Therefore, $a^2 + b^2 = c^2$ and so $b = \sqrt{c^2 - a^2}$.

How much less does the back tire travel? The ratio of the length of the paths of the small back-tire circle to the large front-tire circle is $\sqrt{c^2 - a^2}/c = \sqrt{1 - a^2/c^2}$.

In this situation, a limiting case occurs as the radius of the large circle goes to infinity. Then we can consider the front path to be a straight line, for instance, the x-axis. According to our results, the radius of the back-tire path also goes to infinity, which we should expect, since its path is also a straight line.

We now formulate this in terms of the differential equations; say that the front tire starts at $(c, 0)$. Then, convenient parametric equations are

$$x_f(t) = c\cos(\omega t),\ y_f(t) = c\sin(\omega t).$$

After computing $v(t)$ from formula (3) and simplifying (2), the differential equations for the coordinates of the back contact point become

$$\frac{dx_b(t)}{dt} = (c\omega/a^2)(c\cos(\omega t) - x_b(t))(\sin(\omega t)x_b(t) - \cos(\omega t)y_b(t)) \qquad (6a)$$

$$\frac{dy_b(t)}{dt} = (c\omega/a^2)(c\sin(\omega t) - y_b(t))(\sin(\omega t)x_b(t) - \cos(\omega t)y_b(t)) \qquad (6b)$$

Although equations (6) look difficult to solve, a solution by inspection is readily possible. The previous geometric analysis suggests that the solution should have the form

$$x_b(t) = b\cos(\omega t - \psi) \quad \text{and} \quad y_b(t) = b\sin(\omega t - \psi),$$

where b is the radius, and $\psi = \arcsin(a/c)$ is the phase shift indicating the back tire trails behind the front tire. Insert this trial solution into the differential equation; using standard trigonometric identities you will see that it works, with $b = \sqrt{c^2 - a^2}$.

Simple case III: A stunt circular turn with rear tire as pivot Another limiting case occurs when the radius of the front-tire circle shrinks to a, the wheel-base of the bicycle. In this case, the front tire is turned at a right angle to the bike body and moves in a circle of radius a. According to our formula, the radius of the back-tire path should be 0. This is a stunt turn, a spin or pivot on the back tire. Physically, the back-tire contact point remains motionless.

For this special case, solving the differential equations is easy: Say that the front tire starts at $(a, 0)$, the back tire is positioned at the origin; as in the previous example, write parametric equations for the path of the front tire, and write out the differential equations, with initial conditions $x_b(0) = 0$ and $y_b(0) = 0$. It will then be clear that $x_b(t) = 0$, and $y_b(t) = 0$ satisfy the initial conditions and the differential equations. Therefore, by the uniqueness theorem for first-order ordinary differential equations, this is the unique solution.

When the front path is a sine curve

Set-up and numerical analysis Assume that the front tire follows a sine curve. Experience suggests that the back-tire track should also follow a sine curve with a phase shift and a smaller amplitude.

Start by introducing the path of the front tire, written parametrically as

$$x_f(t) = st,\ y_f(t) = A_f \sin(\xi st). \tag{7}$$

In this set-up, s is a speed parameter that converts time to distance. Let ξ be a spatial frequency, so that one oscillation of the front tire takes $2\pi/\xi$ units of horizontal distance or $2\pi/\xi s$ units of time. Let A_f be the amplitude of the front-tire oscillation. As usual, call a the wheel-base of the bicycle. When we incorporate the parametric equations (7) into the differential equations (2), with an appropriate substitution of the velocity (3), we arrive at the following equations for the x- and y-coordinates of the back tire:

$$\frac{dx_b(t)}{dt} = \left[\frac{s\{st - x_b(t)\}}{a} + \frac{A_f \cos(\xi st)\xi s\{A_f \sin(\xi st) - y_b(t)\}}{a}\right]\frac{st - x_b(t)}{a}. \tag{8}$$

$$\frac{dy_b(t)}{dt} = \left[\frac{s\{st - x_b(t)\}}{a} + \frac{A_f \cos(\xi st)\xi s\{A_f \sin(\xi st) - y_b(t)\}}{a}\right]\frac{A_f \sin(\xi st) - y_b(t)}{a}. \tag{9}$$

We choose $x_b(0) = -a$, $y_b(0) = 0$ as reasonable initial conditions, but other initial conditions are possible.

These equations are certainly difficult to solve analytically; we will first find approximate numerical solutions, for which we need to choose specific values for the parameters. We will take $a = 1$, which, in SI units, means that the wheel-base is 1 meter long; this is reasonable (although just a little short for most adult bicycles, which have a measured wheel-base of slightly more than one meter). Take the amplitude of oscillation A_f to be 0.3 meters, the speed s to be 5 meters per second and the frequency ξ to be 1. The graph of the numerical solution, rescaled to show the details, is shown in FIGURE 2. Of course, the actual path looks far less oscillatory.

The path of the back tire has the general form of a sine curve. Note that the amplitude of oscillation of the back tire is less than the amplitude of oscillation of the

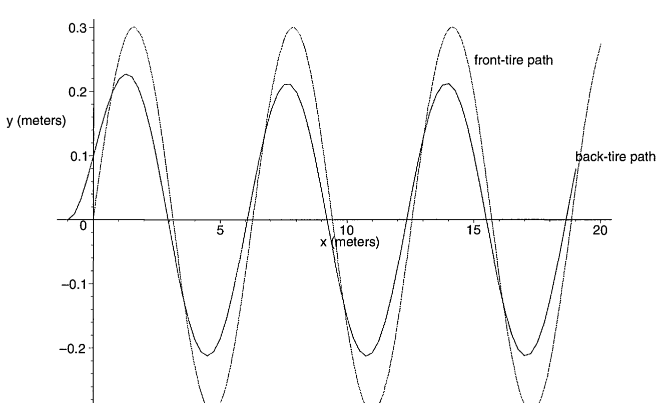

Figure 2 Scaled view of the paths of the front and rear tires, computed numerically with $a = 1$, $A_f = 0.3$, $s = 5$, and $\xi = 1$

front tire, and the back tire is slightly phase-shifted behind the front tire. This seems reasonable.

A good guess at an approximate solution Based on the numerical solution, we can make a good guess at the approximate solution. Since we expect a path similar to the front-tire path, we guess a parametrization, $x_b(t) = st - a$, and $y_b(t) = A_b \sin(\xi st - \psi)$, where A_b is the amplitude of the back-tire path and ψ is a phase shift. Note that this cannot be the exact solution, because here the horizontal distance between the tires' contact points is a in these formulas; in reality, it must be shorter.

From the differential equation (2) (in a slightly different form) we know

$$\frac{y_f(t) - y_b(t)}{x_f(t) - x_b(t)} = \frac{dy_b(t)}{dx_b(t)}.$$

Our assumptions make the left-hand denominator equal to the wheel-base a. Using the chain rule and rearranging the equation, we get

$$y_b(t) = y_f(t) - a\frac{dy_b(t)}{dt}\frac{dt}{dx_b(t)}. \tag{10}$$

Inserting the guessed solution and the known front-tire formula into equation (10) yields

$$A_b \sin(\xi st - \psi) = A_f \sin(\xi st) - a\xi s A_b \cos(\xi st - \psi)\frac{1}{s}.$$

Since this equation must hold for any time, choosing $t = 0$ we get the phase shift

$$\psi = \arctan(\xi a) \tag{11}$$

and using $t = \pi/(2s\xi)$, we get the amplitude

$$A_b = \frac{A_f}{\sqrt{1 + \xi^2 a^2}}.$$

We can plot the guess along with the numerical solution to the equations on scaled axes to compare them in FIGURE 3. Except for a short transient, the guess seems to be identical with the numerical solution. The difference between the numerically

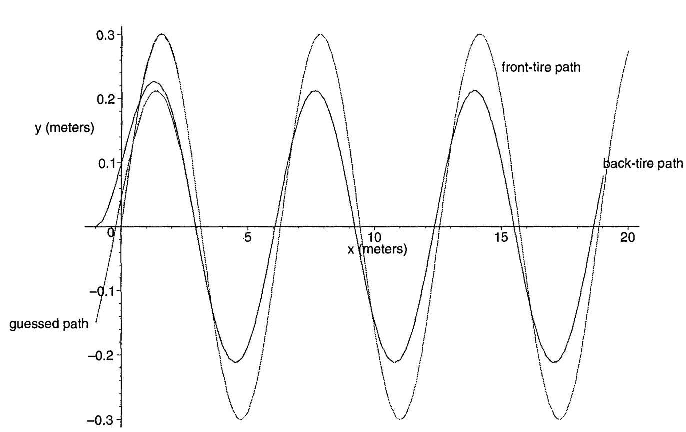

Figure 3 A scaled plot of the front-tire path, the numerically computed back-tire path and the guessed solution

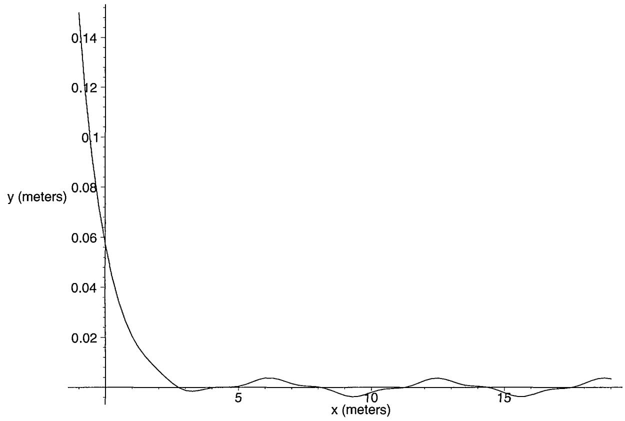

Figure 4 The difference between the numerically computed back-tire path and the sinusoidal guess

computed back-tire path and the guessed solution is plotted in FIGURE 4; except for the transient, the difference is about a centimeter, less than 5% of the magnitude of the back-tire oscillation. The solution $A_b \sin(\xi st - \psi)$ was a very good guess, and the back-tire path is nearly a sine curve. Is there another way to justify the guess?

A solution based on linearized equations The amplitude of the front-tire oscillation is small compared to the other physical parameters. Furthermore, as the amplitude of the front-tire path goes to zero, that is, the path approaches a straight line, the back-tire path approaches the same straight line. We can see that the back-tire path depends on the front-tire amplitude. This suggests that we take the amplitude of the front-tire oscillation to be a small parameter in the differential equations.

We will assume that the back-tire path can be expressed as a power series based on this parameter. Inserting the power series expansion into the differential equations and gathering like terms gives a sequence of linear differential equations that we can solve. In applied mathematics, this method is called a regular perturbation expansion. Regular perturbation is routinely used in all applications where we need to solve a nonlinear equation, at least approximately.

To work out the details, assume

$$x_b(t) = x_{b0}(t) + A_f x_{b1}(t) + \mathcal{O}(A_f^2), \tag{12a}$$

$$y_b(t) = y_{b0}(t) + A_f y_{b1}(t) + \mathcal{O}(A_f^2), \tag{12b}$$

and insert these expansions into equations (8) and (9) to get a perturbation expansion. It is not necessary, but using a symbolic computation system simplifies matters.

Zeroth order Inserting the posited form of the solution, expanding and comparing the terms with no coefficient of A_f, we find an equation for the leading order term

$$\frac{dx_{b0}(t)}{dt} = \frac{s^3t^2}{a^2} - 2\frac{s^2tx_{b0}(t)}{a^2} + \frac{s\,\{x_{b0}(t)\}^2}{a^2} \tag{13}$$

with initial condition $x_{b0}(0) = -a$. Note that this is a Riccati equation because of the term $x_{b0}(t)^2$. However, the right-hand side can be easily factored and the equation solved as either a separable equation or by inspection (or checked!) to yield

$$x_{b0}(t) = st - a. \tag{14}$$

Likewise we can find the leading order equation for $y_{b0}(t)$, using the new information from (14)

$$\frac{dy_{b0}(t)}{dt} = -\frac{sy_{b0}(t)}{a}, \tag{15}$$

with initial condition $y_{b0}(0) = 0$. Because this equation is linear and homogeneous with 0 as initial condition, the solution must be identically zero:

$$y_{b0}(t) = 0. \tag{16}$$

This proves the zeroth order perturbation solution agrees with the formulas in Simple Case I. To lowest order of approximation, the motion of the back tire following a front tire weaving back and forth with small amplitude is a straight line.

First order We equate the terms of the x_b equation with coefficient A_f, and insert the now known solutions (14) and (16), to find

$$\frac{dx_{b1}(t)}{dt} = -\frac{2sx_{b1}(t)}{a}, \tag{17}$$

with initial condition $x_{b1}(0) = 0$. The result is a homogeneous linear differential equation, and is therefore easy to solve:

$$x_{b1}(t) = 0. \tag{18}$$

The equation for $y_{b1}(t)$ is more interesting. Comparing terms with one power of A_f, and using all of the previous information about x_{b0}, y_{b0} and x_{b1}, we find

$$\frac{dy_{b1}(t)}{dt} = \frac{s}{a}\left[\sin(\xi st) - y_{b1}(t)\right]. \tag{19}$$

The solution with initial condition $y_{b1}(0) = 0$ is

$$y_{b1}(t) \doteq \frac{-a\xi\cos(\xi st) + a\xi e^{(-\frac{st}{a})} + \sin(\xi st)}{1 + \xi^2 a^2}.$$

It is easy to use standard identities to write this solution as

$$y_{b1}(t) = \frac{\sin(\xi st - \psi)}{\sqrt{1 + \xi^2 a^2}} + \frac{\sin(\psi)e^{-\frac{st}{a}}}{\sqrt{1 + \xi^2 a^2}},$$

with ψ as in (11).

We now assemble the power series expansion of the solution by substituting $x_{b0}(t)$, $x_{b1}(t)$, $y_{b1}(t)$, and $y_{b2}(t)$ into (12). Ignoring quadratic and higher orders of the front-tire oscillation amplitude, we have the following parametric equations for the motion of the back tire:

$$x_b(t) = st - a \tag{20a}$$

$$y_b(t) = A_f\left[\frac{\sin(\xi st - \psi)}{\sqrt{1 + \xi^2 a^2}} + \frac{\sin(\psi)e^{-\frac{st}{a}}}{\sqrt{1 + \xi^2 a^2}}\right] \tag{20b}$$

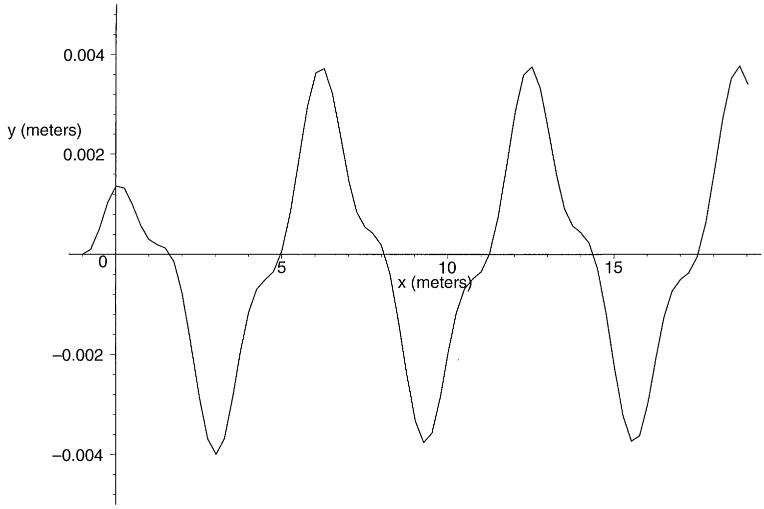

Figure 5 The difference between the numerically computed and the perturbation solutions

We could do some additional work to compare terms with coefficients A_f^2, using the known $x_{b0}(t)$, $x_{b1}(t)$, $y_{b0}(t)$, and $y_{b1}(t)$ to derive linear equations for $x_{b2}(t)$ and $y_{b2}(t)$. However, before we do that let's stop and examine graphically what we have so far.

Plotting the perturbation solutions (20) parametrically, we discover that the back path appears identical with the numerical solution, even including the short transient. In fact, superimposing the perturbation solution with the numerical solution simply yields another copy of FIGURE 2. The difference of the numerically computed back-tire path and the perturbation solution is plotted in FIGURE 5. The transient difference is nearly zero, and the difference is less than 2% of the amplitude of the back-tire oscillation. The back-tire path is very nearly a sine curve with an exponential transient. This explains why the guess was a good approximation except for the transient.

As mentioned above, with more work we could derive additional terms in the perturbation expansion to get an even better approximation. However, it appears that we now have a solution that sufficiently explains and confirms our intuition about the back tire.

A general front-tire path

Now assume that the front-tire path is given by $x_f(t) = st$ and $y_f(t) = A_f\ \mathrm{f}(st)$, where s is a speed parameter and $\mathrm{f}(\cdot)$ is a bounded continuously differentiable function whose maximum value is scaled to be 1. Then the amplitude of the motion of the front tire is a small parameter A_f. Additionally we assume that $\mathrm{f}(0) = 0$ so the motion of the front tire starts at the origin. Again we will discover an approximation for the motion of the back tire by means of regular perturbation expansion.

Looking back at the work in the previous section, we see that the expansions for $x_{b0}(t)$, $y_{b0}(t)$ and $x_{b1}(t)$ don't involve A_f or the front-tire function f. Therefore, the equations and their solutions will be the same, yielding again $x_{b0}(t) = s\,t - a$, $y_{b0}(t) = 0$, and $x_{b1}(t) = 0$. The equation corresponding to (19) for the first-order term $y_{b1}(t)$ using all of this information simplifies nicely to

$$\frac{dy_{b1}(t)}{dt} = \frac{s\mathrm{f}(st)}{a} - \frac{sy_{b1}(t)}{a}$$

with initial condition $y_{b1}(0) = 0$. The solution can be written

$$y_{b1}(t) = \int_0^t \frac{s\mathrm{f}(su)}{a} e^{\frac{-s(t-u)}{a}} \, du.$$

Estimating this integral shows that $y_{b1}(t)$ is bounded by st/a, since $\mathrm{f}(st)$ is bounded by 1. We can get a better bound by doing some deeper analysis of the integral, which is a convolution.

Introduce the variable $w = \frac{s}{a}(t-u)$, and find that

$$y_{b1}(t) = \int_0^{\frac{s}{a}t} \mathrm{f}(st - aw) e^{-w} \, dw.$$

Then let $\mathrm{g}_t(w) = \mathrm{f}(st - aw) I_{[0,(s/a)t]}(w)$, where $I_{[0,S]}(w)$ is the indicator function on $[0, S]$, which is 1 for $w \in [0, S]$ and 0 otherwise. Note that $\mathrm{g}_t(w)$ is continuous, since as $w \to st/a$ then $\mathrm{f}(st - aw) \to \mathrm{f}(0) = 0$ by the assumption that $\mathrm{f}(\cdot)$ was C^1 and starts at the origin ($\mathrm{f}(0) = 0$). Also $\mathrm{g}_t(w)$ is bounded by the bound on $\mathrm{f}(\cdot)$, which we have assumed to be 1. With this notation

$$y_{b1}(t) = \int_0^{\infty} \mathrm{g}_t(w) e^{-w} \, dw,$$

and the nature of the function as a Laplace transform is clearly revealed. Then

$$\begin{aligned} |y_{b1}(t)| &= \left| \int_0^{\infty} \mathrm{g}_t(w) e^{-w} \, dw \right| \\ &\le \int_0^{\infty} |\mathrm{g}_t(w)| e^{-w} \, dw \\ &\le \int_0^{\infty} e^{-w} \, dw = 1. \end{aligned}$$

This means that the amplitude of the back-tire motion never exceeds the amplitude of the front-tire motion, a reasonable conclusion.

Another formulation and solution by iteration

Formulation and diagrams For a nonparametric front-tire path, given as $y_f = \mathrm{f}(x_f)$, an interesting alternative differential equation for the back-tire path results. Of course, any such front-tire path could be put in parametric form and studied using methods from the previous sections. However, looking at the bicycle equations in this new form provides a simple derivation of an unusual form of nonlinear delay-differential equation, interesting in its own right. The method of successive approximations is a typical way to solve nonlinear equations; the bicycle equation provides a nontrivial example in a situation where we have an answer to check against. A good principle of research in applied mathematics is to solve a problem in two different ways and compare the answers. Fortunately for us, both solution methods yield the same results!

As before, take a to be the wheel-base of the bicycle, and assume the front tire starts at the origin, so $\mathrm{f}(0) = 0$. Then the back tire starts at the coordinate $(-a, 0)$. Let the path of the back tire be given as a function $\mathrm{g}(x_b)$ of the x-coordinate of the back tire, x_b. Then we know for example that $\mathrm{g}(-a) = 0$.

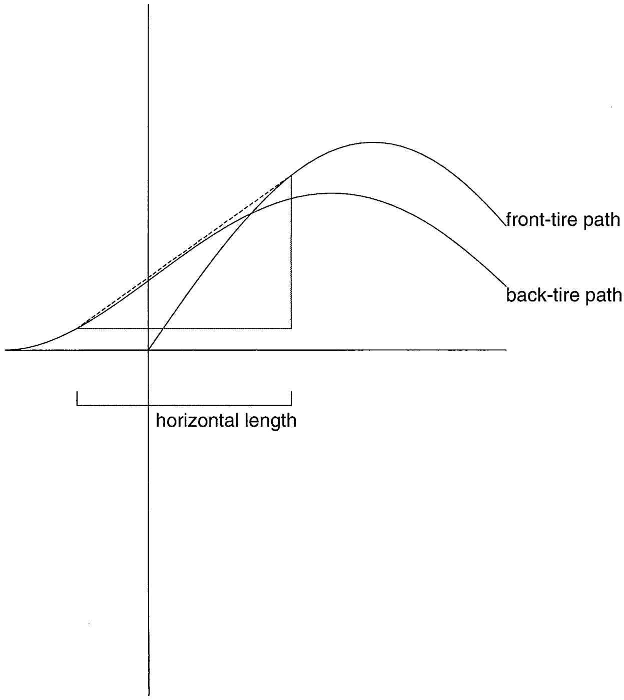

Figure 6 A schematic diagram of the front-tire path, the back-tire path, the bicycle spanning the paths (dashed line), and the projection of the bicycle along the axis giving the horizontal length

We use our notation to express the horizontal distance between the front and back tires, as seen in FIGURE 6:

$$x_f - x_b = \sqrt{a^2 - \left\{\mathrm{f}(x_f) - \mathrm{g}(x_b)\right\}^2}.$$

This gives an implicit relationship between x_f and x_b, assuming that we know the back-tire path $\mathrm{g}(x_b)$. Using the fundamental fact that the tangent vector to the back-tire path points in the direction of the bicycle, we can write

$$\frac{d\mathrm{g}(x_b)}{dx_b} = \frac{\mathrm{f}(x_f) - \mathrm{g}(x_b)}{\sqrt{a^2 - \left\{\mathrm{f}(x_f) - \mathrm{g}(x_b)\right\}^2}}.$$

Again, remember that x_f is given implicitly in terms of x_b, so that there is really only one independent variable x_b in the differential equation. We can rewrite this equation slightly more transparently if we set $x_f = x_b + \Delta$. Then the differential equation is

$$\frac{d\mathrm{g}(x_b)}{dx_b} = \frac{\mathrm{f}(x_b + \Delta) - \mathrm{g}(x_b)}{\sqrt{a^2 - \{\mathrm{f}(x_b + \Delta) - \mathrm{g}(x_b)\}^2}}. \tag{21}$$

Although the differential equation (21) now looks more familiar, it still hides quite a bit of difficulty. Since the right side contains not only the unknown function $\mathrm{g}(x_b)$, but also an advanced argument on the right side, $x_b + \Delta$, this might seem to be a simple delay-differential equation. But this hides an important fact: the delay itself depends on the unknown function $\mathrm{g}(x_b)$ through the implicit relationship $x_f = x_b + \sqrt{a^2 - \{\mathrm{f}(x_f) - \mathrm{g}(x_b)\}^2}$. Therefore, this differential equation has the unknown function appearing not only nonlinearly on the right side, but nonlinearly in the argument of the right side too!

The method of successive approximations is a typical way to solve such an equation. First, approximate the horizontal distance Δ between the front- and rear-tire contact points of the bicycle by the wheel-base a. Generally, this will be an overestimate for the horizontal distance, but it will be close if the amplitude of the paths is not large.

Equation (21) then becomes

$$\frac{dg(x_b)}{dx_b} = \frac{f(x_b + a) - g(x_b)}{\sqrt{a^2 - \{f(x_b + a) - g(x_b)\}^2}}. \tag{22}$$

For a given front-tire path and a known wheel-base this equation is easy to solve numerically.

The next approximation replaces the horizontal distance Δ with

$$\sqrt{a^2 - \{f(x_b + a) - g(x_b)\}^2}.$$

This will also be larger than the true horizontal distance, but clearly a closer approximation. The equation becomes

$$\frac{dg(x_b)}{dx_b} = \frac{f\left(x_b + \sqrt{a^2 - \{f(x_b + a) - g(x_b)\}^2}\right) - g(x_b)}{\sqrt{a^2 - \left\{f(x_b + \sqrt{a^2 - \{f(x_b + a) - g(x_b)\}^2}) - g(x_b)\right\}^2}}. \tag{23}$$

Now the differential equation is clearly more complicated since already the unknown function appears in the argument of the right-hand side as well as nonlinearly. Nevertheless, we can still solve the differential equation numerically. Conceptually, one could use a simple scheme such as the Euler method: Given an initial condition such as $g(-a) = 0$, and knowing the given function $f(x_f)$, the value of the right hand side can be calculated to give the slope of $g(x_b)$ at $x_b = -a$. Then $g(-a + h)$ can be estimated, and the process can be repeated. In practice, of course one uses a more sophisticated technique such as a Runge-Kutta method, or a multi-step predictor-corrector method.

Numerical computation of the back-tire path for comparison As an illustration, we use the iteration procedure to calculate the path of the back tire when the front-tire path is a sine curve as before. Then we can compare the numerical solution of the coupled Riccati equations for the parametric form with the solution calculated numerically by the iteration procedure described above.

We use $f(x_f) = A_f \sin(x_b)$, with front-wheel oscillation amplitude $A_f = 0.3$ and wheel-base $a = 1$ (the same parameters as before). Substituting this information into (22), we find an equation that we can solve numerically.

Plotting the numerical solution along with the zeroth order iteration solution gives a figure that, in print, is virtually indistinguishable from FIGURE 2. For a color version that shows the slight difference, see http://www.maa.org/pubs/mathmag.html. Already, the zeroth order application of this method is almost good enough to fool the eye.

For better results, we can solve the differential equation numerically from the first-order approximation. We can even take one more step in the iteration process to obtain the complicated equation

$$\frac{g(x_b)}{dx_b} = \frac{0.3\sin\left[x_b + \sqrt{1^2 - \left[0.3\sin\left(x_b + \sqrt{1^2 - \{0.3\sin(x_b+1) - g(x_b)\}^2}\right) - g(x_b)\right]^2}\,\right] - g(x_b)}{\sqrt{1^2 - \left\{0.3\sin(x_b + \sqrt{1^2 - \left\{0.3\sin(x_b + \sqrt{1^2 - \{0.3\sin(x_b+1) - g(x_b)\}^2}) - g(x_b)\right\}^2}) - g(x_b)\right\}^2}}.$$

Despite its complicated appearence, the equation can still be solved numerically, with an expected improvement in results. For the purposes of comparison, we present in TABLE 1 all the different solutions for the case where the front tire follows a sine curve. This gives a comprehensive view of the quality of each of the approximate solutions.

TABLE 1: Comparison of the various solutions to the numerical solution. The rows are approximations at various x_b-values, the columns are the various solution methods. The methods labeled g_0, g_1, and g_2 are, respectively, the zeroth, first, and second order iteration methods. The values are rounded to 5 decimal places.

x_b	Numerical Sol.	Perturbation Sol.	Iteration Sol. g_0	Iteration Sol. g_1	Iteration Sol. g_2
0	0.09998	0.10036	0.10097	0.09995	0.09997
1	0.21934	0.21912	0.21986	0.21932	0.21934
2	0.17756	0.17713	0.17720	0.17757	0.17756
3	−0.01191	−0.01273	−0.01520	−0.01174	−0.01192
4	−0.18601	−0.18538	−0.18780	−0.18593	−0.18601
5	−0.18603	−0.18557	−0.18642	−0.18600	−0.18603
6	−0.01541	−0.01440	−0.01268	−0.01555	−0.01540
7	0.17077	0.17028	0.17277	0.17068	0.17077
8	0.19890	0.19851	0.19947	0.19887	0.19890
9	0.04531	0.04426	0.04296	0.04544	0.04531
10	−0.15100	−0.15066	−0.15355	−0.15090	−0.15100

Conclusions We have shown that if the path of the front tire of a bicycle is specified, it is possible to derive the corresponding path of the back tire. In some geometrically simple cases, such as a large circular path for the front tire, it is possible to derive the corresponding back-tire path precisely. In some other reasonable geometric cases, such as a sinusoidal front-tire path, it is not possible to find the corresponding back-tire path precisely, but we can derive approximations to any desired degree of accuracy. In this paper, we have solved the approximation equations to first order, which seems sufficient for most purposes. In fact, the approximation techniques are easy to apply for any reasonably general front-tire path. The only limit to being able to express the solution analytically is the ability to evaluate a convolution, or equivalently to solve a first-order linear differential equation. Of course, in any case, the equations for the back-tire path can be solved numerically.

The coupled nonlinear differential equations for the back-tire path are easy to express and fairly easy to solve when the front-tire path is given parametrically. When the front-tire path is given directly as a function of the position down the road, the differential equations assume the more challenging form of an unusual delay-differential equation, where the delay even depends on the solution. Nevertheless, the problem can still be handled through successive approximations. The solutions found numerically, whether by regular perturbation or successive approximations all agree, and with about the same amount of work, so the choice of technique should be determined by the available information or purpose of the solution.

Our analysis sheds light on the relative distances traveled by the front and back tires in special cases. If the front-tire path is a large circle, the back tire follows a concentric circle, and should experience less wear than the front tire, because the ratio of the circumferences is $\sqrt{1 - a^2/c^2}$. Likewise, if the front tire weaves back and forth along a sine curve, with an amplitude A_f and spatial frequency ξ, then the back tire also follows a sine curve with the smaller amplitude $A_f/\sqrt{1 + a^2\xi^2}$. Although it is not possible to express the arc length of a sine curve with a simple analytic expression, the proportionality of the expressions for the functions show that the arc length traveled by the back tire is proportionately less than that traveled by the front tire. Of course, if the path is perfectly straight, both tires go the same distance.

Can we verify the folklore that on a long bike trip the back-tire wear is less than that of the front tire? Probably not, even though the analysis in this article supports the folklore. Too many other variables intervene in the reality to be modeled so simply. For example, if the back-tire inflation is less than the front tire's, it will wear more. The style of riding, including braking, sliding, and skidding, can affect the wear too.

However, we do offer two somewhat practical consequences from the solutions. First, presented with two intertwined sinusoidal functions, known to be the paths of the front and back tire of a bike, we can now confidently know that the path with the larger amplitude is the front tire, and the path with the proportionally smaller amplitude is the back-tire path. With additional inspection, knowing that the tangent vectors from the back-tire point with fixed distance to the front-tire track, we can find which way the bicycle went. Second, the solutions of the general front-tire path case show that the amplitude of the back-tire path never exceeds the amplitude of the front-tire path, that is, in this model the bike doesn't "fish-tail."

This bicycle problem shows that moderately complicated nonlinear differential equations can be found even in simple everyday experiences. Better yet, we were able to apply several different techniques, yielding solutions of various kinds, giving better understanding of both the everyday experiences and the techniques.

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