Mediterr. J. Math. (2017) 14:9
DOI 10.1007/s00009-016-0815-4
1660-5446/17/010001-19
published online December 27, 2016

Mediterranean Journal of Mathematics

Optimization of Projectile Motion Under Air Resistance Quadratic in Speed

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Abstract. This article centers around the problem of maximizing the horizontal range of a projectile that is launched from atop a tower and is subject only to gravity and air resistance quadratic in speed. Here the surface to which the projectile is launched is represented by a convex impact function, while the projectile motion is described by a classical approximation model for flight curves that is widely considered acceptable for quadratic drag and launch angles up to moderate size. In this setting, the optimal range is given by the point where the impact function intersects the enveloping function induced by the family of flight paths. In the special case of a linear impact function, manageable explicit formulas for the range function, the maximal range, and the corresponding optimal launch angle are provided in terms of the Lambert W function. The article concludes with a solution to Tartaglia's inverse problem in this context.

Mathematics Subject Classification. Primary 26A06; Secondary 49K15.

Keywords. Projectile ballistics, Quadratic drag, Optimal launch angle, Enveloping function, Lambert W function, Tartaglia's inverse problem.

1. Introduction and Motivation

1.1. Historical Background and State of the Art

A projectile in flight near the surface of the Earth has long provided a setting ripe with physical questions and mathematical problems—both abstract and applied. The archetypical case is, of course, that of shooting a projectile from ground level to ground level in a nonresistive medium where the only acting force is gravity. In this context, as early as the seventeenth century, the work of Galileo Galilei revealed that the trajectory of the projectile is parabolic and that the range of the projectile is maximal precisely when it is launched with the angle of elevation $\pi/4$. Galileo also noticed that the inverse problem of determining the angles of inclination leading to a given submaximal range has exactly two solutions, a fact that had been enunciated much earlier by Niccolò Tartaglia; see [4] for a lucid historical discussion and further

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references. Similar optimization problems have been considered in more general settings for quite some time. Indeed, there are ongoing efforts to fashion models that are increasingly refined, comprehensive, and descriptive—both theoretically and practically. Here we only mention the modest case of firing a projectile from atop a tower to ground level or to an inclined plane; see [3,9,11]. Moreover, considerable attention has also been given to the case of projectile motion under linear air resistance; see [3,4,10,13,15].

1.2. Specific Previous Results

As documented in [12], a remarkable amount of information about such optimization problems is available in the case of projectile motion without air resistance. In fact, some of these results even extend to the case of linear air resistance where the retarding force for the projectile is assumed to be proportional to the velocity vector with a negative constant factor. For example, if a projectile is launched to a surface that is represented by a convex impact function, then, in the cases of either no or linear drag, there exists a unique angle of inclination for the launch of the projectile that yields maximal lateral displacement, and, in both cases, this optimal launch angle is characterized by the condition that the tangent lines to the trajectory of the projectile at the launch point and at the point of impact are perpendicular; see Theorem 1.1 and Corollary 4.1 of [12] and Theorem 1 of [13]. In particular, for the case of shooting from ground level to a linear impact function without air resistance, it turns out that the tangent line for optimal launch bisects the angle between the impact line and the vertical through the launch point, a result already known to Halley [9]. On the other hand, for the case of shooting to ground level under linear air resistance, explicit formulas for the optimal launch angle and the maximal lateral displacement were obtained in [10] and [15] based on the Lambert W function. However, although it was observed even by Isaac Newton that it would be much more realistic to consider projectile motion under air resistance quadratic in speed, we are not aware of any previous general optimization result in this direction.

1.3. Main Results of the Present Paper

In this article, we resurrect the classical optimization problem of maximizing the lateral range of a projectile, now, with a humble nod to Newton, in the presence of air resistance that is quadratic in speed. While it is easy to derive from Newton's law the differential equations for the motion of a projectile governed by gravity and quadratic drag, no explicit solution formulas are known. To overcome this difficulty, here we follow a long tradition in this field and work with certain approximations to the exact solutions that are commonly considered acceptable for projectile motion with quadratic drag and small launch angle of inclination; see [7,8,14,16]. Our principal result for this model is Theorem 4.3 in which we analyze the maximal distance, the corresponding optimal launch angle, and the associated flight time in the general case of launching to an arbitrary convex impact function. For the special case of shooting to a linear impact function, manageable closed form solution formulas for the range, the maximal range, and the optimal launch angle in terms of the Lambert W function are provided in Theorem 3.1 and Corollary 4.5. In particular, for launching a projectile from ground level to a linear impact function, we obtain the striking fact that, in our model, the optimal angle for the case of quadratic drag coincides with the one found by Halley for motion without drag; see [6,9,12]. Finally, a general solution to Tartaglia's inverse problem for the case of projectile motion under quadratic drag is presented in Theorem 5.2.

1.4. Outline of the Work

In Sect. 2, we collect the background material for projectile motion with air resistance quadratic in speed and introduce the distance function that results from shooting the projectile to a convex impact function. Then, in Sect. 3, we recall the basics of the Lambert W function which continues to be a powerful and indispensable tool in this context. Here we employ this function to compute the distance function corresponding to a linear impact function. Our main optimization results are contained in Sect. 4. While classical work in this area is often based on a critical point argument or a Lagrange multiplier approach, in this article we use the method of the enveloping function in the spirit of our previous papers [11–13]. The key observation is that the maximal range for our projectile motion coincides with the unique point at which the convex impact function intersects the enveloping function implemented by the family of flight trajectories. In the case of a linear impact function and launch from ground level, this point can be expressed in terms of the Lambert W function. In Sect. 5, we obtain further information about the shape of the graph of the distance function for quadratic drag motion. In particular, again for the special case of shooting from ground level to a linear impact function, we discover that, in our model, the distance function corresponding to quadratic drag exhibits the same kind of symmetry that is known from the classical Galilean model without any air resistance. A summary of our work is provided in Sect. 6.

2. Projectile Motion Under Quadratic Drag

We first develop the equations that describe the motion of a projectile in the xy-plane with air resistance quadratic in speed. Given a tower of height $h > 0$, we suppose that the projectile is launched at time $t = 0$ with initial position $(0, h)$, muzzle speed $s > 0$, and angle of inclination θ with respect to the horizontal line $y = h$ in the direction of the positive x-axis.

Throughout we suppose that the motion of the projectile is governed by two forces. One of these forces is, of course, gravity in the direction of the negative y-axis which results in acceleration of magnitude $g > 0$. The other force is air resistance that here we model by a retarding force for the projectile that is proportional to the square of the speed and points in the direction of the negative velocity vector. Specifically, if $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ denotes the position vector of the projectile at time $t \geq 0$, then we suppose that the retarding force is induced by an acceleration vector of the form

$$-\alpha\,|\mathbf{r}'(t)|^2\,\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\alpha\sqrt{x'(t)^2+y'(t)^2}\,\langle x'(t),y'(t)\rangle\,,$$

where $\alpha > 0$ stands for the quadratic air resistance coefficient.

By Newton's law, this setting leads to the initial value problem

$$\begin{aligned} x''(t) &= -\alpha\sqrt{x'(t)^2+y'(t)^2}\,x'(t), \quad x'(0)=s\cos(\theta), \quad x(0)=0,\\ y''(t) &= -g-\alpha\sqrt{x'(t)^2+y'(t)^2}\,y'(t), \quad y'(0)=s\sin(\theta), \quad y(0)=h. \end{aligned}$$

As indicated earlier, it was observed already by Newton that this quadratic model is often more realistic than the corresponding much more tractable case of air resistance linear in speed, but unfortunately no explicit formulas for the solution $(x(t), y(t))$ of the preceding system of differential equations are known; see, for instance, Chapter 3 of [3] for motion under non-linear drag and, in particular, [7,8] for an excellent exposition of this theory including its long history and further references.

As suggested by [7,8], and also by [14,16], here we choose to work with certain approximations to the exact solutions which are commonly viewed as acceptable for small angles of inclinations, namely

$$\begin{aligned} x_{\mathrm{s}}(t) &= \frac{1}{\alpha}\ln\left(\alpha s\cos(\theta)t+1\right) \quad \text{and}\\ y_{\mathrm{s}}(t) &= h+\left(\frac{\tan(\theta)}{\alpha}+\frac{g\sec^2(\theta)}{2\alpha^2 s^2}\right)\ln(\alpha s\cos(\theta)t+1)-\frac{gt^2}{4}-\frac{gt\sec(\theta)}{2\alpha s}; \end{aligned}$$

see, for instance, formulas (10) and (11) of [8] together with their lucid derivation and discussion. In particular, from Theorem 2 of both [7,8] it is known that $y(t) < y_{\mathrm{s}}(t)$ for all $t > 0$.

It turns out to be beneficial to eliminate the time parameter t in the preceding equations and thus to express the flight curves of our approximation framework as functions of the variable x. For arbitrary $\theta \in (-\pi/2, \pi/2)$, it follows that, in our model, the trajectory traced by the projectile for the launch angle θ is a segment of the graph of the function

$$y = p_\theta(x) = h + \tan(\theta)x - \frac{g\kappa(x)}{4\alpha^2 s^2}\sec^2(\theta), \tag{1}$$

where we use the shortcut notation

$$\kappa(x) = \mathrm{e}^{2\alpha x} - 1 - 2\alpha x = \sum_{k=2}^{\infty}\frac{(2\alpha x)^k}{k!}$$

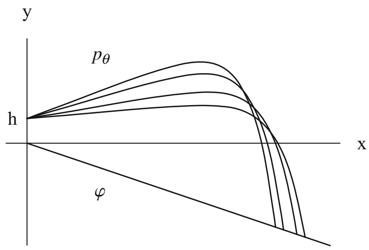
for all $x \geq 0$. Clearly, $\kappa(0) = 0$ and $\kappa(x) > 0$ for all $x > 0$. In the special case $h = 0$, formula (1) reduces to (19) and (12) of [7,8], respectively, which were also derived in [14,16] with different methods. Although (1) represents only an approximation to the solution of our quadratic initial value problem, it has the advantage of being almost as accessible as the corresponding explicit formula for the case of linear air resistance; see, for example, (2) of [13]. In fact, it will be shown that (1) leads to remarkably simple answers to several natural questions.

It is gratifying to see that, by l'Hospital's rule, passing to the limit in (1) as $\alpha \to 0^+$ leads to the standard formula for motion without any air resistance, namely

$$y = h + \tan(\theta)x - \frac{gx^2}{2s^2}\sec^2(\theta); \tag{2}$$

see [12] for a recent discussion of this classical case in the spirit of the present paper.

From now on we consider a projectile whose flight is represented by (1) and which remains airborne until it strikes a surface that is modeled by the graph of a continuous convex function φ on $[0,\infty)$ for which $\varphi(0) = 0$, called the *impact function.* We refer to [17] for the definition of general convex functions and their properties; in particular, by Theorem 4.43 of [17], the convexity of φ already guarantees its continuity on $(0,\infty)$. As indicated by the following graphics for the case of a linear impact function $\varphi(x) = mx$ for some real slope m, our main interest is in comparing the trajectories of the projectile for different angles of inclination θ.



Evidently, we have $\kappa(x)/x \to \infty$ and hence $p_\theta(x)/x \to -\infty$ as $x \to \infty$ for each $\theta \in (-\pi/2, \pi/2)$. Moreover, p_θ is strictly concave, since

$$p''_\theta(x) = -\frac{g}{s^2}\mathrm{e}^{2\alpha x}\sec^2(\theta) < 0$$

for all $x \geq 0$. Because of our conditions on the impact function φ, we may then conclude from Proposition 2.1 of [12] that there exists exactly one $x > 0$ for which $p_\theta(x) = \varphi(x)$. We denote this x by $d(\theta)$, and put $d(\pm\pi/2) = 0$. The main goal is to maximize the resulting *distance function* d, also known as the *range function*, over the interval $[-\pi/2, \pi/2]$. Our approach will allow us to include the classical case $h = 0$ and $\varphi(x) = mx$ for real m when switching to the canonical interval $[\arctan(m), \pi/2]$. Of course, in this case, we have $d(\theta) > 0$ only when $\arctan(m) < \theta < \pi/2$.

While, in general, there is no hope for an explicit formula for the distance function d induced by an arbitrary impact function φ, in Sect. 3 we will provide a closed form for the solution x of the equation $p_\theta(x) = \varphi(x)$ in the case of a linear impact function φ. Section 4 will address the even harder problem of maximizing the distance function d over $[-\pi/2, \pi/2]$, also known as *the optimal range problem.*

3. The Range for Quadratic Drag Motion and Linear Impact

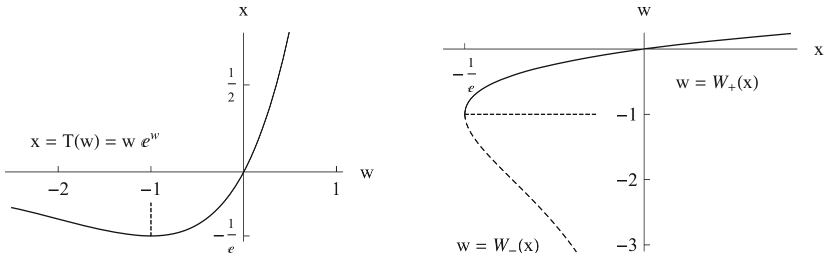
In this section, we consider the motion of a projectile corresponding to (1) together with the impact function $\varphi(x) = mx$ for an arbitrary real constant

m. Inspired by results in [10,15] for the case of linear drag and $m = 0$, we are interested in an explicit formula for the corresponding range in terms of the Lambert W function. This classical function originates from the function T given by $T(w) = w\mathrm{e}^w$ for all $w \in \mathbb{R}$.

As suggested by the following graphics and confirmed by basic calculus, T maps the real line onto the interval $[-1/\mathrm{e}, \infty)$ and is strictly decreasing on $(-\infty, -1]$ and strictly increasing on $[-1, \infty)$. Let

$$W_+ : [-\frac{1}{\mathrm{e}}, \infty) \to [-1, \infty) \quad \text{and} \quad W_- : [-\frac{1}{\mathrm{e}}, 0) \to (-\infty, -1]$$

denote the inverses of T when restricted to the intervals $[-1, \infty)$ and $(-\infty, -1]$, respectively.



W_+ is also known as the main branch of the Lambert W function and denoted by ProductLog in *Mathematica*, while *Maple* uses just LambertW. In Theorem 5 of [10], this branch was employed to obtain an explicit formula for the optimal range of a projectile when shooting to ground level $\varphi = 0$ under linear air resistance, after the special case $h = 0$ was settled in Theorem 3 of [15]; see also [18]. In the present context, the second branch W_- will also play an important role.

It will be convenient to consider the non-dimensional parameter $\gamma = g/(\alpha s^2) > 0$ as well as the function ζ given by

$$\zeta(\theta) = \sin(2\theta) - 2m\cos^2(\theta) + \gamma$$

for all $\theta \in [-\pi/2, \pi/2]$. Note that $\zeta(\theta) = 2\,(\tan(\theta) - m)\cos^2(\theta) + \gamma$ for $\theta \neq \pm\pi/2$.

Theorem 3.1. *The distance function d corresponding to the projectile motion represented by (1) and the impact function $\varphi(x) = mx$ for an arbitrary real m is given by the following formulas: for arbitrary $\theta \in [-\pi/2, \pi/2]$, we have*

$$d(\theta) = \frac{1}{2\alpha}\left(-\frac{4\alpha h\cos^2(\theta) + \gamma}{\zeta(\theta)} - W_+\left(-\frac{\gamma}{\zeta(\theta)}\exp\left(-\frac{4\alpha h\cos^2(\theta) + \gamma}{\zeta(\theta)}\right)\right)\right)$$

provided that $\zeta(\theta) < 0$ and

$$d(\theta) = \frac{1}{2\alpha}\left(-\frac{4\alpha h\cos^2(\theta) + \gamma}{\zeta(\theta)} - W_-\left(-\frac{\gamma}{\zeta(\theta)}\exp\left(-\frac{4\alpha h\cos^2(\theta) + \gamma}{\zeta(\theta)}\right)\right)\right)$$

provided that $\zeta(\theta) > 0$, *while*

$$d(\theta) = \frac{1}{2\alpha} \ln\left(1 + \frac{4\alpha h \cos^2(\theta)}{\gamma}\right)$$

in the case $\zeta(\theta) = 0$. *Finally, the flight time corresponding to a launch angle* θ *in* $(-\pi/2, \pi/2)$ *is given by*

$$t(\theta) = \frac{1}{\alpha s \cos(\theta)} \left(\mathrm{e}^{\alpha d(\theta)} - 1\right).$$

Proof. Since $\zeta(\pm\pi/2) = \gamma > 0$ and $W_-(-1/\mathrm{e}) = -1$, the second of the displayed formulas correctly reflects the fact that $d(\pm\pi/2) = 0$. It remains to consider the case when $-\pi/2 < \theta < \pi/2$. We already know that the equation $p_\theta(x) = mx$ has a unique solution $x > 0$, namely $x = d(\theta)$. To obtain a formula for x, we rewrite this equation in the form

$$h + (\tan(\theta) - m)\, x - \frac{g}{4\alpha^2 s^2} \left(\mathrm{e}^{2\alpha x} - 1 - 2\alpha x\right) \sec^2(\theta) = 0,$$

which is equivalent to

$$a + b2\alpha x + c\mathrm{e}^{2\alpha x} = 0 \tag{3}$$

with the choice

$$a = h + \frac{\gamma}{4\alpha} \sec^2(\theta), \quad b = \frac{2\,(\tan(\theta) - m) + \gamma \sec^2(\theta)}{4\alpha}, \quad c = -\frac{\gamma}{4\alpha} \sec^2(\theta).$$

Clearly, $a > 0$ and $c < 0$, while $b = \zeta(\theta) \sec^2(\theta)/(4\alpha)$ changes sign with $\zeta(\theta)$. In particular, $\zeta(\theta) = 0$ holds precisely when $b = 0$, and, in this case, Eq. (3) assumes the simple form $a + c\mathrm{e}^{2\alpha x} = 0$ with the obvious solution

$$x = \frac{1}{2\alpha} \ln\left(-\frac{a}{c}\right) = \frac{1}{2\alpha} \ln\left(1 + \frac{4\alpha h \cos^2(\theta)}{\gamma}\right),$$

as stipulated.

Since *Mathematica* has a problem with the two branches of the Lambert W function when attempting to solve Eq. (3) for the case $b \neq 0$, it is helpful to have a closer look at the function ξ given by $\xi(x) = a + b2\alpha x + c\mathrm{e}^{2\alpha x}$ for all real x, not just for $x > 0$. Note that $\xi(0) = h > 0$, while $\xi'(x) = 2\alpha\left(b + c\mathrm{e}^{2\alpha x}\right)$ and $\xi''(x) = 4\alpha^2 c\mathrm{e}^{2\alpha x} < 0$ for all $x \in \mathbb{R}$. If $b < 0$, then ξ is both strictly decreasing and strictly concave on $(-\infty, \infty)$ with $\xi(x) \to -\infty$ as $x \to \infty$, so that (3) has exactly one real solution which has to be strictly positive. However, if $b > 0$, then the strictly concave function ξ satisfies $\xi(x) \to -\infty$ as $x \to \infty$ and also as $x \to -\infty$, so that (3) has precisely two real solutions, one of them in $(0, \infty)$ and the other one in $(-\infty, 0)$.

Now, for $b \neq 0$, we observe that (3) is equivalent to

$$-\frac{a}{b} - 2\alpha x = \frac{c}{b} \mathrm{e}^{2\alpha x}$$

and therefore to

$$\left(-\frac{a}{b} - 2\alpha x\right) \exp\left(-\frac{a}{b} - 2\alpha x\right) = \frac{c}{b} \exp\left(-\frac{a}{b}\right).$$

But this means precisely that

$$T\left(-\frac{a}{b} - 2\alpha x\right) = \frac{c}{b} \exp\left(-\frac{a}{b}\right) \tag{4}$$

and thus allows us to put the Lambert W function into action.

If $\zeta(\theta) < 0$ and hence $b < 0$, then the right-hand side of Eq. (4) is strictly positive which means that only the main branch of the Lambert W function is relevant here. Consequently, in the case $\zeta(\theta) < 0$, we obtain

$$-\frac{a}{b} - 2\alpha x = W_+\left(\frac{c}{b}\exp\left(-\frac{a}{b}\right)\right)$$

and therefore

$$x = \frac{1}{2\alpha}\left(-\frac{a}{b} - W_+\left(\frac{c}{b}\exp\left(-\frac{a}{b}\right)\right)\right).$$

On the other hand, if $\zeta(\theta) > 0$ and hence $b > 0$, then we know that (4) has exactly two real solutions, one of them given by W_+ and the other one by W_-. Since we are interested in the larger of the two solutions, here we need to take the second branch of the Lambert W function. Thus, in the case $\zeta(x) > 0$, we obtain

$$x = \frac{1}{2\alpha}\left(-\frac{a}{b} - W_-\left(\frac{c}{b}\exp\left(-\frac{a}{b}\right)\right)\right).$$

The definition of a, b, and c then leads to the desired formulas for $d(\theta)$ for the case $\zeta(\theta) \neq 0$. The final claim for the flight time $t(\theta)$ is a simple consequence of the identities $\alpha d(\theta) = \alpha x_{\mathrm{s}}(t(\theta)) = \ln(\alpha s\cos(\theta)t(\theta) + 1)$. $\square$

It may not be immediate from the three formulas for $d(\theta)$ provided in Theorem 3.1 that the distance function d is continuous on $[-\pi/2, \pi/2]$, but this will be established in Theorem 5.2 even in a much more general setting.

A glance at the preceding proof reveals that Theorem 3.1 remains valid in the case $h = 0$ once θ is restricted to $[\arctan(m), \pi/2]$. In fact, only the formula involving the second branch of the Lambert W function is needed to express the range of the projectile in the case $h = 0$, since, for all θ in $[\arctan(m), \pi/2]$, we have $\zeta(\theta) \geq \gamma > 0$ and therefore

$$d(\theta) = \frac{1}{2\alpha}\left(-\frac{\gamma}{\zeta(\theta)} - W_-\left(-\frac{\gamma}{\zeta(\theta)}\exp\left(-\frac{\gamma}{\zeta(\theta)}\right)\right)\right). \tag{5}$$

Since $W_+(we^w) = W_+T(w) = w$ for all $w \geq -1$, it is clear that the corresponding formula with W_- replaced by W_+ would result in $d(\theta) = 0$, the trivial second solution of the equation $p_\theta(x) = mx$ in the case $h = 0$. Unfortunately, *Mathematica* ignores the subtle difference between W_+ and W_- for equations of this type. For the classical case of shooting from ground level to ground level, i.e., $h = 0$ and $m = 0$, an equivalent version of formula (5) was recently established in [1].

4. Maximal Range and Optimal Launch Angle

Sparked by the success of the enveloping function method for projectile motion with no or linear air resistance in [11–13], our approach to the optimal range problem for quadratic drag will be based on the solution of the equation $\psi(c) = \varphi(c)$ for $c > 0$, where φ is the impact function in place and ψ denotes the *enveloping function* of the family of flight functions p_θ from formula (1). Thus ψ is defined by

$$\psi(x) = \sup\{p_\theta(x) : \theta \in (-\pi/2, \pi/2)\}$$

for all $x > 0$, while, correspondingly, $\psi(0)$ is defined as the maximal height of the projectile in the air when launched with the angle $\theta = \pi/2$.

The latter exceptional case may be handled as follows: first one takes the limit as $\theta \to \pi/2^-$ in our previous formula for $y_s(t)$ for fixed $t \geq 0$, and then one computes the maximum value of the resulting height function. Elementary calculus leads to $\psi(0) = h + s^2/(2g)$, as the interested reader may easily check. More importantly, a manageable formula for ψ on $(0, \infty)$ is provided in the following result.

Lemma 4.1. *Consider the projectile motion determined by* (1) *with corresponding enveloping function* ψ*. Then, for arbitrary fixed* $x > 0$*, the height function* ρ_x *given by* $\rho_x(\theta) = p_\theta(x)$ *for all* $\theta \in (-\pi/2, \pi/2)$ *satisfies* $\rho_x(\theta) \to -\infty$ *as* $\theta \to -\pi/2^+$ *and as* $\theta \to \pi/2^-$*. Moreover,* ρ_x *has the unique global maximum point*

$$\theta_x = \arctan\left(\frac{2\alpha x}{\gamma\kappa(x)}\right) > 0$$

and is strictly increasing on $(-\pi/2, \theta_x)$ *and strictly decreasing on* $(\theta_x, \pi/2)$*. Finally, the maximum value is*

$$\psi(x) = \rho_x(\theta_x) = h + x\tan(\theta_x) - \frac{\gamma\kappa(x)}{4\alpha}\left(1 + \tan^2(\theta_x)\right) = h + \frac{\alpha x^2}{\gamma\kappa(x)} - \frac{\gamma\kappa(x)}{4\alpha}.$$

Proof. Because $\sec^2(\theta) = 1 + \tan^2(\theta)$ and $\gamma = g/(\alpha s^2)$, formula (1) leads to

$$\rho_x(\theta) = h + \tan(\theta)x - \frac{\gamma\kappa(x)}{4\alpha}\left(1 + \tan^2(\theta)\right)$$

for all $\theta \in (-\pi/2, \pi/2)$. From this the desired convergence is immediate. Moreover, we obtain

$$\rho_x'(\theta) = \sec^2(\theta)x - \frac{\gamma\kappa(x)}{2\alpha}\tan(\theta)\sec^2(\theta) = \sec^2(\theta)\left(x - \frac{\gamma\kappa(x)}{2\alpha}\tan(\theta)\right)$$

for all $\theta \in (-\pi/2, \pi/2)$. By basic calculus, this establishes the second claim, and the computation of $\psi(x)$ is then straightforward. $\square$

As a consequence of Lemma 4.1, a simple application of l'Hospital's rule confirms that $\psi(x)$ converges to the anticipated limit $\psi(0) = h + s^2/(2g)$ as $x \to 0^+$. Thus ψ is continuous on the entire interval $[0, \infty)$, as one would expect. Further properties of ψ are exhibited in the following result.

Proposition 4.2. *The enveloping function* ψ *corresponding to* (1) *satisfies*

$$\psi'(x) = p_{\theta_x}'(x) < -\cot(\theta_x) < 0$$

for all $x > 0$*. In particular,* ψ *is strictly decreasing.*

Proof. Again, because $\sec^2(\theta) = 1+\tan^2(\theta)$ and $\gamma = g/(\alpha s^2)$, we obtain from (1) that

$$p'_{\theta_x}(x) = \eta(x) - \frac{\gamma\kappa'(x)}{4\alpha}\left(1+\eta^2(x)\right)$$

with the choice $\eta(x) = \tan(\theta_x)$. On the other hand, we know from Lemma 4.1 that

$$\psi(x) = h + x\eta(x) - \frac{\gamma\kappa(x)}{4\alpha}\left(1+\eta^2(x)\right)$$

which implies, by the product rule, that

$$\psi'(x) = \eta(x) + x\eta'(x) - \frac{\gamma\kappa(x)}{4\alpha}2\eta(x)\eta'(x) - \frac{\gamma\kappa'(x)}{4\alpha}\left(1+\eta^2(x)\right).$$

The two middle terms on the right-hand side cancel, since Lemma 4.1 confirms that $\gamma\kappa(x)\eta(x) = 2\alpha x$. Thus $p'_{\theta_x}(x) = \psi'(x)$. Moreover, it follows that

$$\begin{aligned}
\psi'(x) + \cot(\theta_x) &= \eta(x) - \frac{\gamma\kappa'(x)}{4\alpha}\left(1+\eta^2(x)\right) + \frac{1}{\eta(x)} \\
&= \frac{1+\eta^2(x)}{\eta(x)}\left(1 - \frac{\gamma}{4\alpha}\kappa'(x)\eta(x)\right) \\
&= \frac{1+\eta^2(x)}{\eta(x)\kappa(x)}\left(\kappa(x) - \frac{\gamma}{4\alpha}\kappa'(x)\eta(x)\kappa(x)\right) \\
&= \frac{1+\eta^2(x)}{\eta(x)\kappa(x)}\left((1-\alpha x)\mathrm{e}^{2\alpha x} - 1 - \alpha x\right),
\end{aligned}$$

where we recycled the fact that $\gamma\kappa(x)\eta(x) = 2\alpha x$. Consequently, it remains to be seen that the function δ given by $\delta(x) = (1-x)\mathrm{e}^{2x} - 1 - x$ satisfies $\delta(x) < 0$ for all $x > 0$. For this we observe that

$$\delta'(x) = 2(1-x)\mathrm{e}^{2x} - \mathrm{e}^{2x} - 1 = \mathrm{e}^{2x} - 2x\mathrm{e}^{2x} - 1$$

and therefore

$$\delta''(x) = 2\mathrm{e}^{2x} - 2\mathrm{e}^{2x} - 4x\mathrm{e}^{2x} = -4x\mathrm{e}^{2x} < 0$$

for all $x > 0$. We conclude that δ' strictly decreases on $(0,\infty)$ and satisfies $\delta'(0) = 0$ which ensures that $\delta'(x) < 0$ for all $x > 0$. Therefore, δ is strictly decreasing on $(0,\infty)$. Because $\delta(0) = 0$, it follows that $\delta < 0$ on $(0,\infty)$, as desired. □

A few remarks are in order. Although θ_x was created to satisfy merely $\psi(x) = p_{\theta_x}(x)$, the corresponding identity for the derivatives is not really surprising. In fact, the result carries over from the cases of no or linear drag and could be derived from the construction of the enveloping function as in the proof of Proposition 4 of [13].

However, in the cases of no or linear drag, the identity $p'_{\theta_x}(x) = -\cot(\theta_x)$ holds for all $x > 0$, which turns out to be the clue to a remarkable geometric characterization of the optimal launch angle in these two cases. Indeed, for arbitrary $\theta \in (-\pi/2, \pi/2)$, the tangent lines to the flight function p_θ for no or linear drag at the launch point and at the point of impact are perpendicular precisely when θ provides maximal range; see Corollary 4.1 of [12] and Theorem 1 of [13]. It now follows from Proposition 4.2 and Theorem 4.3 below that this characterization ceases to be true for our quadratic drag model, probably reflecting the fact that here we only work with certain approximations.

Moreover, for no or linear drag, the enveloping function is strictly concave, but this does not extend to the present context. Indeed, this is illustrated by the graph of ψ corresponding to the data $g = 9.81\ \mathrm{m/s}^2$, $h = 0$ m, $s = 200$ m/s, and $\alpha = .015\ \mathrm{m}^{-1}$, as the reader may easily verify.

We proceed with our main optimization result.

Theorem 4.3. *Let ψ be the enveloping function corresponding to (1), and consider an arbitrary continuous convex impact function φ on $[0, \infty)$ with $\varphi(0) = 0$. Then there exist a unique point $c > 0$ for which $\psi(c) = \varphi(c)$ and a unique global maximum point θ_{opt} in $[-\pi/2, \pi/2]$ for the distance function d resulting from φ. Moreover, the maximal distance is exactly $c = d(\theta_{\mathrm{opt}})$, while the optimal angle may be obtained from c by the formula*

$$\theta_{\mathrm{opt}} = \theta_c = \arctan\left(\frac{2\alpha c}{\gamma\kappa(c)}\right) > 0.$$

Finally, the flight time for the optimal launch is given by

$$t_{\mathrm{opt}} = \frac{\mathrm{e}^{\alpha c} - 1}{\alpha s \gamma \kappa(c)} \sqrt{\gamma^2\kappa^2(c) + 4\alpha^2 c^2}.$$

Proof. We first show that the equation $\psi(c) = \varphi(c)$ has at least one solution $c > 0$. Since φ is convex and satisfies $\varphi(0) = 0$, for arbitrary $u \geq 1$ we obtain

$$\varphi(1) = \varphi\left(\frac{1}{u}u + \left(1 - \frac{1}{u}\right)0\right) \leq \frac{1}{u}\varphi(u) + \left(1 - \frac{1}{u}\right)\varphi(0) = \frac{1}{u}\varphi(u)$$

and therefore $\varphi(1)u \leq \varphi(u)$. On the other hand, from $\kappa(u)/u \to \infty$ as $u \to \infty$ we infer that

$$\frac{\psi(u)}{u} = \frac{h}{u} + \frac{\alpha u}{\gamma\kappa(u)} - \frac{\gamma\kappa(u)}{4\alpha u} \to -\infty \quad \text{as} \quad u \to \infty.$$

Consequently, for all sufficiently large $u \geq 1$, we obtain $\psi(u) < \varphi(1)u \leq \varphi(u)$ and hence $\psi(u) - \varphi(u) < 0$. Because $\psi(0) - \varphi(0) > h > 0$, the intermediate value theorem then ensures that there exists some $c > 0$ for which $\psi(c) - \varphi(c) = 0$, as desired.

Now, given an arbitrary point $c > 0$ for which $\psi(c) = \varphi(c)$, Lemma 4.1 entails that $p_{\theta_c}(c) = \rho_c(\theta_c) = \psi(c) = \varphi(c)$ and therefore $d(\theta_c) = c$. Moreover, for each $\theta \in (-\pi/2, \pi/2)$ with $\theta \neq \theta_c$, Lemma 4.1 shows that

$$p_\theta(c) = \rho_c(\theta) < \rho_c(\theta_c) = \psi(c) = \varphi(c).$$

Because $p_\theta(0) = h > 0 = \varphi(0)$, the intermediate value theorem yields $p_\theta(x) = \varphi(x)$ for some $x \in (0, c)$ and hence $d(\theta) < c = d(\theta_c)$. Consequently, θ_c is the unique global maximum point of d, and the maximal distance is c. In particular, this establishes the uniqueness of the solution $c > 0$ of $\psi(c) = \varphi(c)$. Finally, because

$$\sec(\theta_c) = \sqrt{1 + \tan^2(\theta_c)} = \sqrt{1 + \frac{4\alpha^2 c^2}{\gamma^2\kappa^2(c)}},$$

the last formula for the flight time may be easily deduced from

$$c = d(\theta_c) = x_s\left(t_{\text{opt}}\right) = \frac{1}{\alpha}\ln\left(\alpha s\cos(\theta_c)t_{\text{opt}} + 1\right)$$

by solving for t_{opt}. □

Theorem 4.3 leads to interesting information about the maximal range and optimal launch angle as functions of the given parameters. As a sample, we note the following consequence which complements corresponding results from [10,11] for no and linear drag.

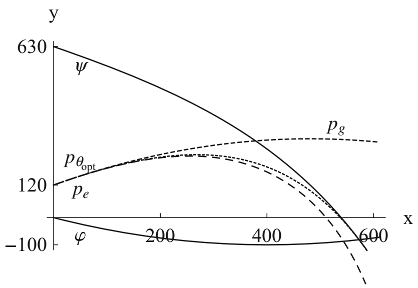
Corollary 4.4. *In the setting of Theorem 4.3, the maximal distance $d(\theta_{\text{opt}})$ is a strictly increasing function of the initial height h when all the other parameters are kept fixed, while the optimal angle θ_{opt} is strictly decreasing in h under this condition.*

Proof. The first assertion follows easily from the fact that $d(\theta_{\text{opt}})$ is the unique solution $c > 0$ of the equation $\psi(c) = \varphi(c)$. By the formula for θ_{opt} provided in Theorem 4.3, it remains to be seen that the function χ given by $\chi(c) = c/\kappa(c)$ for all $c > 0$ is strictly decreasing on $(0, \infty)$. But this is immediate from

$$\chi'(c) = \frac{(1 - 2\alpha c)\mathrm{e}^{2\alpha c} - 1}{\kappa^2(c)} = \frac{1}{\kappa^2(c)}\sum_{k=2}^{\infty}\tfrac{1-k}{k!}(2\alpha c)^k < 0$$

for all $c > 0$. □

We continue with a typical numerical example for Theorem 4.3. Let $g = 9.81$ m/s^2, $h = 120$ m, $s = 100$ m/s, and $\alpha = 0.002$ m^{-1}. The corresponding enveloping function ψ is then determined by Lemma 4.1. For the choice of the impact function φ given by $\varphi(x) = 0.000625(x - 400)^2 - 100$, the FindRoot command of *Mathematica* leads to a numerical solution of the equation $\psi(c) = \varphi(c)$, namely $c = 572.721$ m. By Theorem 4.3, this is an approximation for the maximal range $d\left(\theta_{\text{opt}}\right)$ in this setting, and the formula for the optimal launch angle from this result then leads to $\theta_{\text{opt}} = 0.616359$ radian or 35.3148°. It is now possible to graph the corresponding flight curve together with ψ and φ to geometrically confirm the correctness of this solution. Our figure also includes the Galilean flight parabola p_{g} for dragless motion with $\theta = \theta_{\text{opt}}$ in formula (2), as well as the curve p_{e} given by a numerical representation of the exact solution $(x(t), y(t))$ of our initial value problem, based on the NDSolve command of *Mathematica*.



As predicted by Theorem 2 of [7] and also of [8], the curve p_{e} is dominated by the graph of $p_{\theta_{\mathrm{opt}}}$ which in turn is below the parabola p_{g}. Moreover, the graphics indicate that p_{e} and φ intersect at $\hat{c} = 543.6$ m. Taking for granted that the optimal range problem for the exact solutions of the initial value problems has a maximal solution c_0 implemented by an angle θ_0, we conclude that $\hat{c} \leq c_0 \leq d(\theta_0) \leq d(\theta_{\mathrm{opt}}) = c$. Thus, although the exact value of c_0 remains unknown, it follows that c provides a numerical approximation to c_0 that is accurate within an error of only 5.36%. This illustrates that, in the present example, the model based on $(x_{\mathrm{s}}(t), y_{\mathrm{s}}(t))$ works quite well even though our optimal angle $\theta_{\mathrm{opt}} = 35.3148°$ is not particularly small.

For the classical case of linear impact functions, Theorem 4.3 allows us to develop explicit solution formulas at least when $h = 0$.

Corollary 4.5. *The maximal range for the projectile motion* (1) *when shooting from ground level $h = 0$ to the line $\varphi(x) = mx$ for arbitrary real m is given by*

$$c = d(\theta_{\mathrm{opt}}) = \frac{1}{2\alpha}\left(-\frac{1}{1+\mu} - W_-\left(-\frac{1}{1+\mu}\exp\left(-\frac{1}{1+\mu}\right)\right)\right)$$

with the choice

$$\mu = \frac{\alpha s^2}{g}\left(\sqrt{1+m^2} - m\right) = \frac{\sqrt{1+m^2} - m}{\gamma} > 0.$$

Moreover, the optimal launch angle is

$$\theta_{\mathrm{opt}} = \operatorname{arccot}\left(\sqrt{1+m^2} - m\right) = \arctan\left(\sqrt{1+m^2} + m\right) = \frac{\arctan(m)}{2} + \frac{\pi}{4},$$

while the corresponding flight time may be obtained from the formula provided in Theorem 4.3 *for the preceding value of c.*

Proof. By Theorem 4.3, it suffices to solve the equation $\psi(c) = mc$ for $c > 0$. In the case $h = 0$, the formula for the enveloping function from Lemma 4.1 simplifies to

$$\psi(c) = \frac{\alpha c^2}{\gamma\kappa(c)} - \frac{\gamma\kappa(c)}{4\alpha} = \frac{4\alpha^2c^2 - \gamma^2\kappa^2(c)}{4\alpha\gamma\kappa(c)}.$$

Hence the equation $\psi(c) = mc$ may be rewritten as $4\alpha^2c^2 - \gamma^2\kappa^2(c) = 4\alpha\gamma\kappa(c)mc$ or equivalently as

$$\kappa^2(c) + \frac{4\alpha mc}{\gamma}\kappa(c) - \frac{4\alpha^2c^2}{\gamma^2} = 0.$$

The solutions of this quadratic equation for $\kappa(c)$ are

$$\kappa(c) = -\frac{2\alpha mc}{\gamma} \pm \sqrt{\frac{4\alpha^2m^2c^2}{\gamma^2} + \frac{4\alpha^2c^2}{\gamma^2}} = \frac{2\alpha c}{\gamma}\left(-m \pm \sqrt{1+m^2}\right).$$

Because $c > 0$ and $\kappa(c) > 0$, we conclude that $\psi(c) = mc$ holds precisely when

$$\kappa(c) = 2\alpha c\mu \tag{6}$$

with the constant μ provided in the assertion. To solve the last equation in terms of the Lambert W function, we recycle the technique from our proof of Theorem 3.1 and consider the following equivalences

$$\begin{aligned}\psi(c) = mc &\iff \mathrm{e}^{2\alpha c} - 1 - 2\alpha c = 2\alpha c\mu \\ &\iff -\frac{1}{1+\mu}\mathrm{e}^{2\alpha c} = -2\alpha c - \frac{1}{1+\mu} \\ &\iff -\frac{1}{1+\mu}\exp\left(-\frac{1}{1+\mu}\right) = \left(-2\alpha c - \frac{1}{1+\mu}\right)\exp\left(-2\alpha c - \frac{1}{1+\mu}\right) \\ &\iff T\left(-\frac{1}{1+\mu}\right) = T\left(-2\alpha c - \frac{1}{1+\mu}\right),\end{aligned}$$

where, as before, $T(w) = w\mathrm{e}^w$ for all $w \in \mathbb{R}$. From the graph of T we know that

$$-2\alpha c - \frac{1}{1+\mu} \leq -1 < -\frac{1}{1+\mu} < 0$$

which leads to the equation

$$-2\alpha c - \frac{1}{1+\mu} = W_-\left(-\frac{1}{1+\mu}\exp\left(-\frac{1}{1+\mu}\right)\right)$$

and hence the desired formula for c. Moreover, since we have seen in (6) that $\kappa(c) = 2\alpha c\mu$, the definition of μ yields

$$\frac{2\alpha c}{\gamma\kappa(c)} = \frac{1}{\sqrt{1+m^2} - m} = \sqrt{1+m^2} + m.$$

Hence the first two equalities in the stipulated formula for θ_{opt} and also the final assertion are clear from Theorem 4.3. It remains to be seen that

$$\arctan\left(\sqrt{1+m^2} + m\right) = \frac{\arctan(m)}{2} + \frac{\pi}{4}$$

for all real m. Since a simple computation confirms that both sides have the same derivative with respect to m, the claim follows from the fact that the formula holds for $m = 0$. Alternatively, as shown in Section 5 of [12], the equality may also be derived from basic trigonometric identities. $\square$

Thus, somewhat surprisingly, it turns out that, in the setting of Corollary 4.5, the optimal launch angle θ_{opt} depends only on the slope m of the impact line, but not on the parameters g, s, and α. In fact, the last formula for θ_{opt} in Corollary 4.5 shows that the optimal angle is a strictly increasing function of the slope m.

The same formula also reveals that the tangent line to the optimal flight path at the origin bisects the angle between the positive y-axis and the impact line $\varphi(x) = mx$, exactly as in the case of projectile motion without drag, a classical fact that dates back to Halley [9]; see also Section 4 of [6] and Corollary 4.1 of [12].

On the other hand, it is interesting to note that, in Corollary 4.5, the tangent line to the optimal flight path at the point of impact *fails* to bisect the angle between the impact line $\varphi(x) = mx$ and the vertical line through the impact point, since we know from Proposition 4.2 and Theorem 4.3 that the tangent lines to the optimal flight path at the origin and at the impact point are *not* perpendicular. This stands in sharp contrast to the situation of projectile motion without drag where, by Corollary 4.1 of [12], the bisection property at the point of impact also characterizes the optimal flight path.

Furthermore, by Theorem 2 of [11], and also by Section 4.1 of [10], the angle θ_{opt} of Corollary 4.5 coincides with the optimal launch angle for projectile motion in the same setting, but without any air resistance, i.e., the motion given by (2) in the special case $h = 0$. In particular, for shooting from ground level to ground level, we obtain $\theta_{\mathrm{opt}} = \pi/4$, exactly as in the archetypical Galilean case of motion in a nonresistive medium.

Along the same line, we note the following convergence result, which is in the same spirit as Corollary 6 of [10] for the case of air resistance linear in speed.

Corollary 4.6. *For arbitrary $\alpha > 0$, let c_α denote the maximal range for the projectile motion considered in Corollary 4.5, and let c_0 stand for the maximal range for the projectile motion in the same setting, but without any air resistance. Then we have $c_\alpha \to c_0$ as $\alpha \to 0^+$.*

Proof. First, Corollary 4.5 confirms that $2\alpha c_\alpha \to -1 - W_-(-1/\mathrm{e}) = 0$ as $\alpha \to 0^+$. Moreover, from formula (6) of the proof of this result we also know that

$$\kappa(c_\alpha) = 2\alpha c_\alpha \mu_\alpha = 2\alpha c_\alpha \frac{\alpha s^2}{g}\left(\sqrt{1+m^2} - m\right)$$

and therefore

$$2\frac{\mathrm{e}^{2\alpha c_\alpha} - 1 - 2\alpha c_\alpha}{4\alpha^2 c_\alpha^2} = \frac{s^2}{c_\alpha g}\left(\sqrt{1+m^2} - m\right)$$

for all $\alpha > 0$. Here the left-hand side converges to 1 as $\alpha \to 0^+$, since it is well known and easily seen that $2(\mathrm{e}^x - 1 - x)/x^2 \to 1$ as $x \to 0$. From the right-hand side we then conclude that

$$c_\alpha \to \frac{s^2}{g}\left(\sqrt{1+m^2} - m\right) \quad \text{as} \quad \alpha \to 0^+.$$

But this limit is exactly c_0, as shown in Theorem 2 of [11]. $\square$

5. Tartaglia's Inverse Problem for Quadratic Drag Motion

Our main undertaking in this section is the following problem attributable to the Italian mathematician Niccolò Fontana Tartaglia (1499–1557) of Brescia.

Definition 5.1. Tartaglia's inverse problem means to determine the number of angles of inclination for launch that result in a given value for the range of a projectile.

We cast Tartaglia's inverse problem in the general setting of Theorem 4.3 and thus are led to a closer look at the distance function d considered there. By the intermediate value theorem, the following result confirms that, for arbitrary x with $0 < x < d(\theta_{\mathrm{opt}})$, the equation $d(\theta) = x$ has exactly two solutions θ in $(-\pi/2, \pi/2)$, one of them, of course, smaller than θ_{opt} and the other one larger.

For a lucid exposition of the historical background of *Tartaglia's inverse problem* and solutions in the case of shooting from ground level to ground level under linear air resistance, we refer to [4,5,15], while more general impact functions were recently handled in [12,13].

Note that, in our general context, the functions φ and d need not be differentiable. This means that the monotonicity properties of d cannot be established by just checking the sign of the derivative and thus require a more subtle argument from real analysis.

Theorem 5.2. *In the general setting of Theorem 4.3 for quadratic drag, the distance function d is continuous on $[-\pi/2, \pi/2]$, strictly increasing on $[-\pi/2, \theta_{\text{opt}}]$, and strictly decreasing on $[\theta_{\text{opt}}, \pi/2]$.*

Proof. (i) Since Theorem 4.3 entails that d is bounded, to establish the continuity of d it suffices, by Theorem 12.56 of [2], to show that the graph of d is closed in $\mathbb{R}^2$. To this end, we consider an arbitrary sequence $(\theta_k)_{k\in\mathbb{N}}$ in $[-\pi/2, \pi/2]$ that converges to some limit θ such that the sequence $(d(\theta_k))_{k\in\mathbb{N}}$ also converges to some limit x. Our task is to prove that $d(\theta) = x$.

For this, we first consider the case $\theta \neq \pm\pi/2$. Then we have $\theta_k \neq \pm\pi/2$ for almost all $k \in \mathbb{N}$, and for all such k the definition of d confirms that $p_{\theta_k}(d(\theta_k)) = \varphi(d(\theta_k))$ and therefore

$$h + \tan(\theta_k)d(\theta_k) - \frac{g\kappa(d(\theta_k))}{4\alpha^2 s^2}\sec^2(\theta_k) = \varphi(d(\theta_k)).$$

By continuity, passing to the limit as $k \to \infty$ leads to

$$h + \tan(\theta)x - \frac{g\kappa(x)}{4\alpha^2 s^2}\sec^2(\theta) = \varphi(x)$$

and thus $p_\theta(x) = \varphi(x)$. Because $p_\theta(0) = h > 0 = \varphi(0)$, this implies that $x > 0$ and hence $x = d(\theta)$, as desired.

We next turn to the case $\theta = \pi/2$ for which we have to show that $x = 0$. This is obvious when $\theta_k = \pi/2$ for infinitely many $k \in \mathbb{N}$. So we may assume that $\theta_k \neq \pm\pi/2$ for almost all $k \in \mathbb{N}$, and, as before, for all such k the definition of d leads to

$$h + \left(\sin(\theta_k)\cos(\theta_k)d(\theta_k) - \frac{g\kappa(d(\theta_k))}{4\alpha^2 s^2}\right)\sec^2(\theta_k) = \varphi(d(\theta_k)).$$

By continuity, the right-hand side converges as $k \to \infty$ to the real number $\varphi(x)$. It follows that $x = 0$, since otherwise we would have $x > 0$ so that the left-hand side would converge to $-\infty$, because $\kappa\,(d(\theta_k)) \to \kappa(x) > 0$ as $k \to \infty$. This settles the case $\theta = \pi/2$, and the remaining case $\theta = -\pi/2$ may be handled by the same method and is, in fact, even easier.

(ii) To establish the remaining assertions, we recall that a continuous function on a closed bounded interval $[u, v]$ with exactly one local maximum point w in $[u, v]$ strictly increases on $[u, w]$ and strictly decreases on $[w, v]$, a fact that may be deduced from the extreme value theorem, as shown in Lemma 5 of [11].

Hence it remains to be seen that any θ in $[-\pi/2, \pi/2]$ different from θ_{opt} fails to provide a local maximum for d. For $\theta = \pm\pi/2$, this is trivial, while, for $\theta \in (-\pi/2, \pi/2)$, we define $x = d(\theta)$ and consider the following three cases.

If $\theta < \theta_x$, then, for all θ^* with $\theta < \theta^* < \theta_x$, we infer from Lemma 4.1 that

$$\varphi(x) = \varphi(d(\theta)) = p_\theta(d(\theta)) = p_\theta(x) = \rho_x(\theta) < \rho_x(\theta^*) = p_{\theta^*}(x).$$

On the other hand, from the first part of the proof of Theorem 4.3 we know that $p_{\theta^*}(u) \le \psi(u) < \varphi(u)$ for all sufficiently large $u > 0$ which, by the intermediate value theorem, ensures that $p_{\theta^*} < \varphi$ on $(d(\theta^*), \infty)$. Because $\varphi(x) < p_{\theta^*}(x)$, we conclude that $x < d(\theta^*)$. Thus $d(\theta) < d(\theta^*)$ whenever $\theta < \theta^* < \theta_x$, which shows that d has no local maximum at θ.

Similarly, if $\theta_x < \theta$, then it follows that $d(\theta) < d(\theta^*)$ whenever $\theta_x < \theta^* < \theta$, so that d fails to have a local maximum at θ also in the case $\theta_x < \theta$.

Finally, if $\theta_x = \theta$, then $d(\theta_x) = d(\theta) = x$ and hence $\psi(x) = \rho_x(\theta_x) = p_{\theta_x}(x) = \varphi(x)$. By Theorem 4.3, we conclude that $x = d(\theta_{\text{opt}})$ and therefore $\theta = \theta_{\text{opt}}$, as desired. □

Remarkably specific information about the solution of Tartaglia's inverse problem may be gleaned in the special case of launching from ground level $h = 0$ to a linear impact function $\varphi(x) = mx$ for arbitrary real m. In fact, in this case, the graph of the distance function exhibits the same symmetry as in the classical Galilean model without any drag.

Theorem 5.3. *In the setting of Corollary 4.5, we have*

$$d(\theta_{\text{opt}} - \theta) = d(\theta_{\text{opt}} + \theta) \qquad \textit{for all } \theta \in [0, \pi/4 - \arctan(m)/2].$$

Proof. The clue to the argument will be the corresponding identity for ζ together with the distance formula (5) from Sect. 3. Let $\omega = \arctan(m)$. Then we know from Corollary 4.5 that $\theta_{\text{opt}} = \omega/2 + \pi/4$, and it is clear that

$$\cos(\omega) = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin(\omega) = \frac{m}{\sqrt{1+m^2}}.$$

For arbitrary $\theta \in [\omega, \pi/2]$, we conclude from basic trigonometric identities that

$$\begin{aligned}
\sin(2\theta) - 2m\cos^2(\theta) &= \sin(2\theta) - m\cos(2\theta) - m \\
&= \sqrt{1+m^2}\,(\sin(2\theta)\cos(\omega) - \cos(2\theta)\sin(\omega)) - m \\
&= \sqrt{1+m^2}\,\sin(2\theta - \omega) - m.
\end{aligned}$$

Hence, for all $\theta \in [0, \pi/4 - \omega/2]$, the symmetry of the sine function about the vertical line $x = \pi/2$ yields

$$\begin{aligned}
\zeta(\theta_{\text{opt}} - \theta) &= \sqrt{1+m^2}\,\sin(2(\theta_{\text{opt}} - \theta) - \omega) - m + \gamma \\
&= \sqrt{1+m^2}\,\sin(\pi/2 - 2\theta) - m + \gamma \\
&= \sqrt{1+m^2}\,\sin(\pi/2 + 2\theta) - m + \gamma \\
&= \sqrt{1+m^2}\,\sin(2(\theta_{\text{opt}} + \theta) - \omega) - m + \gamma
\end{aligned}$$

$$= \zeta(\theta_{\text{opt}} + \theta).$$

By formula (5), this implies $d(\theta_{\text{opt}} - \theta) = d(\theta_{\text{opt}} + \theta)$, as desired. □

6. Concluding Comments

The present work addresses a natural optimization problem that arises from the scenario of a projectile being launched from atop a tower and confronting quadratic drag during flight. The approach to maximizing the lateral displacement of the projectile entails cultivation of a classical approximation model for projectile motion with air resistance quadratic in speed. As in the cases of no or linear air resistance, the enveloping function induced by the family of flight paths plays a major role: its unique point of intersection with the convex function representing the impact surface is the key to unlocking the maximal range of the projectile, the optimal angle for launch, and the associated flight time.

When the impact function is modeled by a line, the Lambert W function enables manageable explicit formulas for the range function, the maximal range, and the corresponding optimal launch angle. Launching from ground level to a linear impact function exposes a pleasing geometric aspect of the model: the tangent line to the optimal flight path at the launch point bisects the angle between the impact line and the vertical through the launch point, exactly as in the case of projectile motion without drag.

The article closes by confirming various analytic aspects of the distance function in the general setting. Specifically, continuity and detailed information about its monotonicity lead to a solution of Tartaglia's inverse problem for determining the number of launch angles that result in a prescribed range. Here again, the case of launching from ground level to a linear impact function lends extra geometric insight—the graph of the distance function exhibits symmetry that is remarkably akin to that of the classical dragless environment.

As with the case of projectile motion under no or linear air resistance, we hope that our results will be of interest to teachers and students of both real analysis and applied mathematics.

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Received: April 8, 2016.
Revised: September 2, 2016.
Accepted: November 24, 2016.