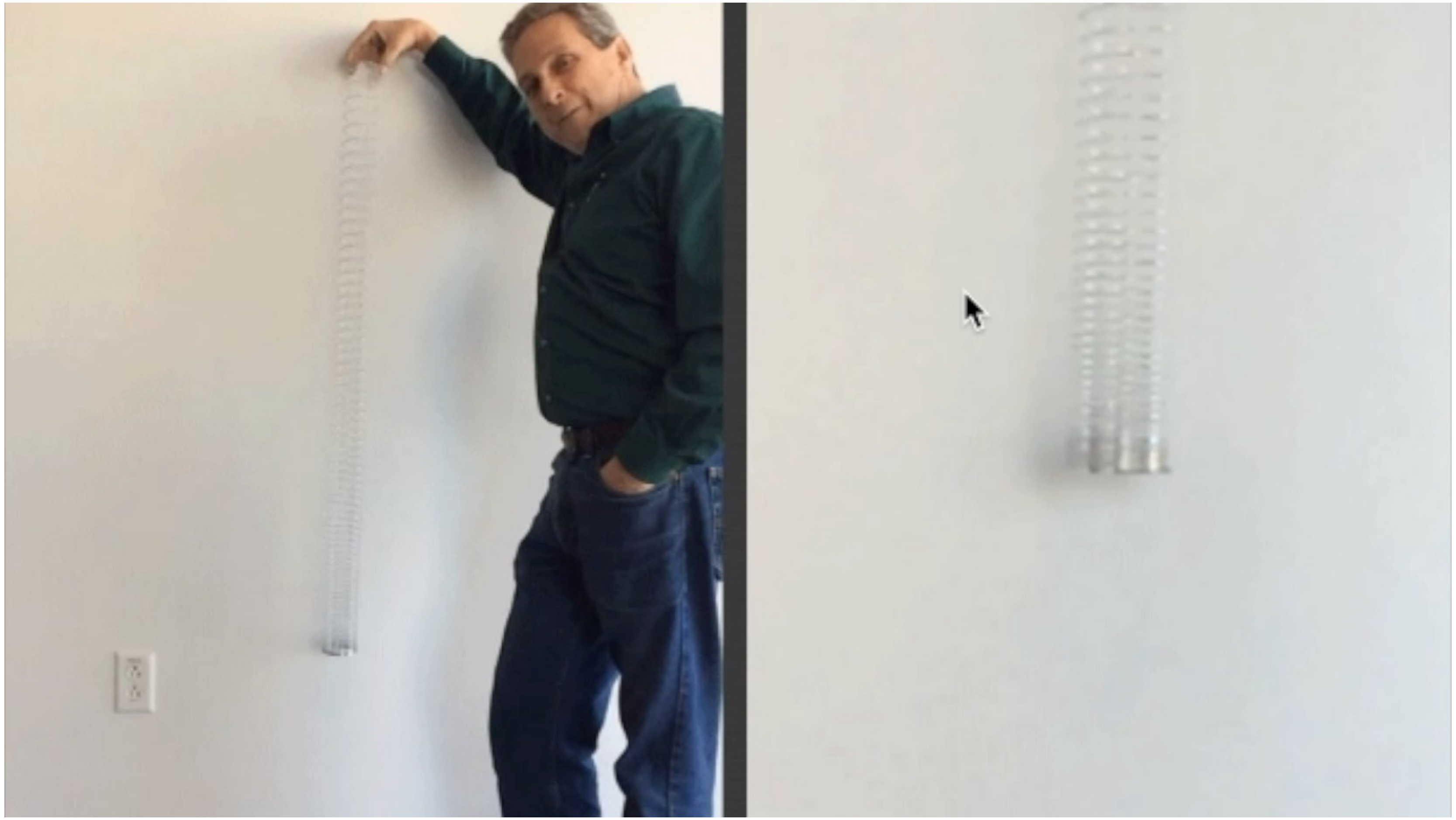
General Idea



Main Goal: Find how long it takes for the slinky to collapse (how long the bottom of the slinky stays still)

Overview

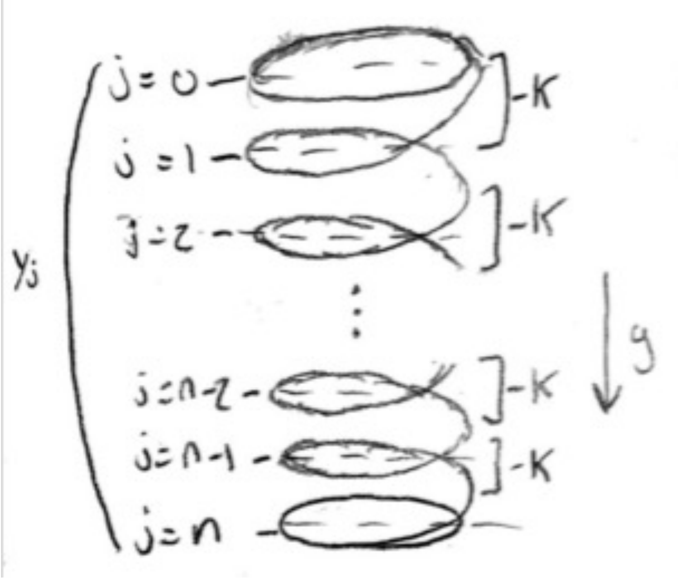
- Discrete Approximation
- Continuum Solution
- Temporal Solution

Discrete Approximation

Forces Present: Spring Force and Gravity

Using Hooke's law we can calculate the spring forces between each position on the slinky

Hooke's Law: $F_{spring} = -\mathrm{k} * \Delta y_j$



For j = 0: $m * \frac{d^2y}{dt^2} = -k(y_0 - y_1)$

j = 1: $m * \frac{d^2y}{dt^2} = -k(y_1 - y_0) - k(y_1 - y_2) = -\mathrm{k}(-y_0 + 2y_1 - y_2)$

j = 2: $m * \frac{d^2y}{dt^2} = -k(y_2 - y_1) - k(y_2 - y_3) = -\mathrm{k}(-y_1 + 2y_2 - y_3)$

j = n-1: $m * \frac{d^2y}{dt^2} = -k(y_{n-1} - y_{n-2}) - k(y_{n-1} - y_n) = -\mathrm{k}(-y_{n-2} + 2y_{n-1} - y_n)$

j = n: $m * \frac{d^2y}{dt^2} = -k(y_n - y_{n-1})$

We also have the force of gravity acting on each coil of the slinky: $m * \frac{d^2y}{dt^2} = -mg$ for j = 0, 1, 2, … , n

Discrete Approximation Continued

Writing the out the equation using Newton's second Law, we have:

For j = 0: $m * \frac{d^2y}{dt^2} = -k(y_0 - y_1)$ - mg

For j = 1: $m * \frac{d^2y}{dt^2} = -\mathrm{k}(-y_0 + 2y_1 - y_2)$ - mg

For j = 2: $m * \frac{d^2y}{dt^2} = -\mathrm{k}(-y_1 + 2y_2 - y_3)$ - mg

For j = n-1: $m * \frac{d^2y}{dt^2} = -\mathrm{k}(-y_{n-2} + 2y_{n-1} - y_n)$ - mg

For j = n: $m * \frac{d^2y}{dt^2} = -k(y_n - y_{n-1})$ - mg

Which we can generalize as: $m\ddot{y} = -kAy - mge$.

$$y = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Static Initial Conditions

No acceleration → $\cancel{m\ddot{y}} = -kAy - mge$ → $kAy = -mge$ → $-Ay = \frac{mge}{k}$

Initial location for the very top of the slinky: $y_0(0) = 0.$

Initial location of the j^th point of the slinky: $y_j(0) = aj + bj^2.$ → The author does not go into detail for how this formula is obtained so do not bother to try to figure it out

Similarly: $y_{j-1}(0) = a(j-1) + b(j-1)^2$

$y_{j+1}(0) = a(j+1) + b(j+1)^2$

<u>Find specific values for a, b, and y_j:</u>

Compute (-)*Ay* $y_{j-1}(0) - 2y_j(0) + y_{j+1}(0) = \frac{mg}{k},$

$y_{n-1}(0) - y_n(0) = \frac{mg}{k}.$

→ $a = -\frac{mg}{k}(n + 1/2).$ $b = \frac{mg}{2k}$

Use a, b, to solve for $y_j(0)$: $y_j(0) = -\frac{mg}{k}\left(nj - \frac{1}{2}j(j-1)\right)$ → This is what we need to take away from the discrete approximation in order to complete the continuum solution

Continuum Solution (limit as n → ∞)

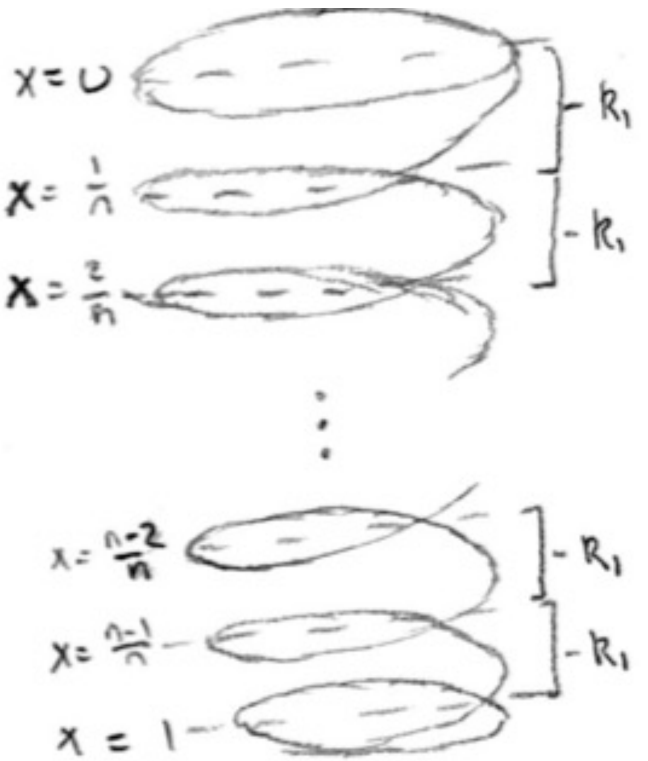
Replace the index j with a continuous variable x where $x = j/n$ → $x = 0$ (top of slinky)
$x = 1$ (bottom of slinky)

From slide 3 $y_j(0) = -\frac{mg}{k}\left(nj - \frac{1}{2}j(j-1)\right)$ using the fact that $x = j/n$ or $j = nx$

it can be shown with effort that $y(x,0) = -\frac{m_1 g}{k_1}\left(n^2 x - \frac{\cancelto{n^2}{n(n-1)}\,x^2}{2}\right)$

$$y(x,0) = -\frac{m_1 g n^2}{k_1}\left(x - \frac{x^2}{2}\right)$$

where m_1 is the mass of one coil and k_1 is the spring constant of one coil



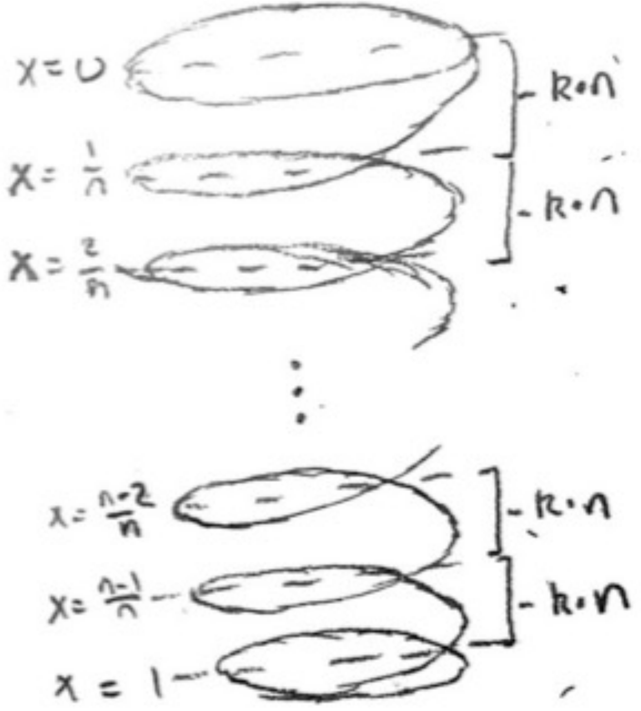
Continuum Solution (limit as n → ∞)

From the previous slides, we had the equations $y(x,0) = -\frac{m_1 g n^2}{k_1}\,(x - \frac{x^2}{2})$ and $m_1\ddot{y} = -k_1 A y - mge$

In the continuum solution, we want to re-write these equations where *m* and *k* are for the entire slinky

- m_1 = mass of 1 coil on the slinky
  m = mass of the entire slinky
  $m_1 = m/n$
- k_1 = spring constant of 1 coil
  k = spring constant for the entire slinky
  - From physics, we get the following relationship between k and k_1

  $k_1 = k * n$



Equation for the initial displacement of each coil:

$$y(x,0) = -\frac{m_1 g n^2}{k_1}\,(x - \frac{x^2}{2}) = -\frac{\frac{m}{n} g n^2}{k*n}\,(x - \frac{x^2}{2}) = -\frac{mg}{k}\,(x - \frac{x^2}{2})$$

Equation obtained from using Newton's second law:

$$m_1\ddot{y} = -k_1 A y - mge \rightarrow \frac{m}{n}\,\ddot{y} + (k*n)Ay = -mge \rightarrow \ddot{y} + \frac{k}{m}\frac{A}{1/n^2}y = -ge$$

Scaling

Newtons second law equation for continuum solution: $\ddot{y} + \frac{k}{m}\frac{A}{1/n^2}y = -ge$

From lecture 11 (where *h = 1/n*), we can see for the scaling of $1/n^2$, $\frac{A}{1/n^2}$ resembles the second derivative matrix, thus as n → ∞ $\frac{A}{1/n^2} \longrightarrow -\frac{\partial^2}{\partial x^2}$

"negative second derivative of height y with respect to x"

First and last rows of A do not resemble $-\frac{\partial^2}{\partial x^2}$

From slide 1: Ay = $\pm(y_{j-1} - y_j)$ → resembles ($\frac{dy}{dx}$) from lecture 9 which leads to the fact that $\frac{\partial y}{\partial x}(x, t) = 0, \quad x \in \{0, 1\}$

Note: Not essential in order to find the time it takes for the slinky to collapse, so do not concern yourself with understanding it.

Continuum Solution Initial Conditions

The general equation for continuum solution needs to satisfy the wave equation, which has the conditions…

Note: in the general equation, k and m are for the entire slinky as discussed on slide 5

$$\frac{\partial^2 y}{\partial t^2}(x,t) - \frac{k}{m}\frac{\partial^2 y}{\partial x^2}(x,t) = -g, \qquad 0 < x < 1,\ t > 0,$$ → Equation of motion

$$y(x,0) = -\frac{mg}{k}\left(x - \frac{1}{2}x^2\right), \qquad 0 \le x \le 1,$$ → Initial displacement y as a function of x (index), from two slides ago

$$\frac{\partial y}{\partial t}(x,0) = 0, \qquad 0 \le x \le 1,$$ → No initial velocity at t = 0

$$\frac{\partial y}{\partial x}(x,t) = 0, \qquad x \in \{0,1\},\ t \ge 0.$$ → Boundary condition

From previous slide

Continuum solution (no initial or boundary conditions yet)

Recall from lecture 6: $y_{General} = y_{Particular} + y_{Homogeneous}$

Particular Solution:

$$y_P(x,t) = -\frac{gt^2}{2} \rightarrow \ddot{y}_P = -g$$

Show that this solution works:

$$\ddot{y}_p + \frac{k}{m} * \frac{\partial^2}{\partial x^2} y_p = -g \rightarrow \cancel{\ddot{y}_p} + \frac{k}{m} * \frac{\partial^2}{\partial x^2} y_p = \cancel{-g} \rightarrow + \frac{k}{m} * \frac{\partial^2}{\partial x^2} y_p = 0$$

Homogeneous Solution:

$$y_H(x,t) = f_+\left(x + \sqrt{k/m}\, t\right) + f_-\left(x - \sqrt{k/m}\, t\right)$$

→ f_+ and f_- are arbitrary functions

→ This form is studied in a graduate course on partial differential equations

General Solution:

$$y(x,t) = f_+\left(x + \sqrt{k/m}\, t\right) + f_-\left(x - \sqrt{k/m}\, t\right) - \frac{1}{2} g t^2$$

Satisfy Initial Conditions

Given the general equation: $y(x,t) = f_+\left(x + \sqrt{k/m}\, t\right) + f_-\left(x - \sqrt{k/m}\, t\right) - \frac{1}{2}gt^2$

Show that $\frac{\partial y}{\partial t}(x,0) = 0$ True because the slinky is released at time t = 0 so there is no velocity

$$\frac{\partial y}{\partial t}(x,t) = \sqrt{\frac{k}{m}} f'_+\left(x + \sqrt{k/m}\, t\right) - \sqrt{\frac{k}{m}} f'_-\left(x - \sqrt{k/m}\, t\right) - gt$$

$$\frac{\partial y}{\partial t}(x,0) = \sqrt{\frac{k}{m}} f'_+(x) - \sqrt{\frac{k}{m}} f'_-(x) = 0$$

What we have found here is that $f'_+(x) = f'_-(x)$ which implies that $f_+(x) = f_-(x) + c$

One can show with effort that c = 0 so $f_+(x) = f_-(x) = f(x)$

Satisfy Initial Conditions

Given the general equation: $y(x,t) = f_+\left(x + \sqrt{k/m}\, t\right) + f_-\left(x - \sqrt{k/m}\, t\right) - \frac{1}{2}gt^2$

Satisfy that $y(x,0) \;=\; -\frac{mg}{k}\left(x - \frac{1}{2}x^2\right)$

At t = 0 in the general equation: $y(x,0) = f_+(x) + f_-(x)$ = *f(x) + f(x) = 2f(x)*

Using the fact that $y(x,0) = -\frac{mg}{k}\left(x - \frac{x^2}{2}\right)$ from slide 4

We can then conclude that *f(x)* = $\frac{y(x,0)}{2} = -\frac{mg}{2k}\left(x - \frac{x^2}{2}\right)$

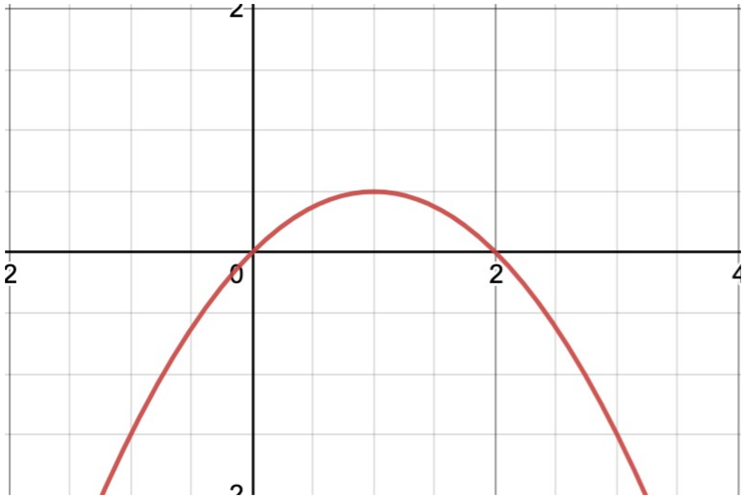
This satisfies the condition that $y(x,0) \;=\; -\frac{mg}{k}\left(x - \frac{1}{2}x^2\right)$

Satisfy the boundary conditions

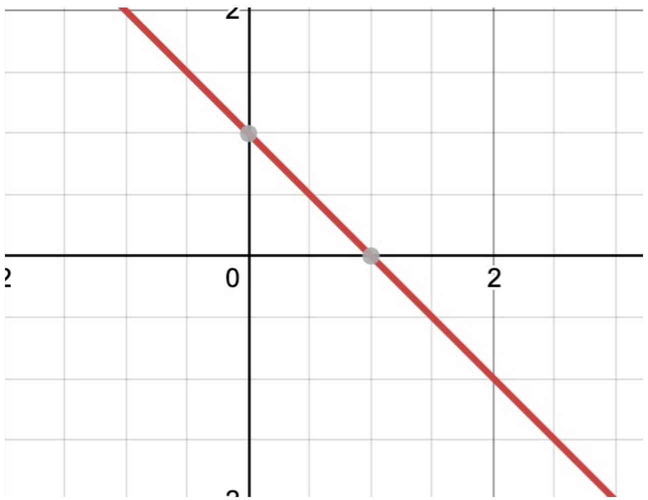
From earlier, we had that $\frac{\partial y}{\partial x}(x, t) = 0$, for x = 0 and x = 1

To satisfy this condition, we need to compute f'(x): f(x) $= -\frac{mg}{2k}\left(x - \frac{x^2}{2}\right)$ → f'(x) = $-\frac{mg}{2k}(1 - x)$

f(x):



f'(x):



- x = 1

  It is easy to see above that at x = 1, the boundary condition is satisfied
- x = 0

  Looking at the graphs, it is not apparent that the boundary condition at x = 0 is satisfied
  It is not clear in the paper how they arrived at this boundary condition for x = 0
  As a result, the professor has informed to to take this boundary condition as given and proceed
  Also note: x is only in the range of 0 to 1

Temporal Solution

Given the formula $y(x,t) = f(x + \sqrt{k/m}\,t) + f(x - \sqrt{k/m}\,t) - \frac{1}{2}gt^2$

The author of the article then specifies that this function f is only valid on the interval [0,2]

Implies that the term $x \pm \sqrt{\frac{k}{m}}\,t$ must be between 0 and 2

Since the index *x* ranges from $0 \le x \le 1$, it can be shown with some effort that $0 \le t \le \sqrt{\frac{m}{k}}\,x$

It can then be shown with some effort that

$$y(x,t) = -\frac{mg}{4k}\left(x + \sqrt{\frac{k}{m}}\,t\right)\left(2 - x - \sqrt{\frac{k}{m}}\,t\right) - \frac{mg}{4k}\left(x - \sqrt{\frac{k}{m}}\,t\right)\left(2 - x + \sqrt{\frac{k}{m}}\,t\right) - \frac{1}{2}gt^2 \qquad 0 \le t \le \sqrt{\frac{m}{k}}\,x, \quad 0 \le x \le 1$$

$$= -\frac{mg}{2k}x(2 - x) = y(x,0)$$ → The equation for the initial displacement for each coil on the slinky

Conclusion:

- each index on the slinky stays still for time t = $\sqrt{\frac{m}{k}}\,x$
- The bottom of the slinky (where x = 1) stays still for time t = $\sqrt{\frac{m}{k}}$

   → Which implies that it takes t = $\sqrt{\frac{m}{k}}$ for the slinky to fully collapse.

Realistic Case:

- The time it takes for a slinky to fully collapse is smaller than t = $\sqrt{\frac{m}{k}}$
- The continuum solution does not consider that the individual coils of the slinky bump into each other while falling.
- This action of bumping into each other will speed up the falling of the slinky.

Summary

<u>Discrete Approximation</u>: Gave us the equation for the initial displacement of each coil on the slinky.

<u>Continuum Solution / Temporal Solution:</u> Combined, these two sections demonstrated how in theory, each coil on the slinky will stay still on the time interval [0, $\sqrt{\frac{m}{k}}x$], thus the time it takes for the slinky to fully collapse is $\sqrt{\frac{m}{k}}$.