Technology:

MATLAB or other high-level numerical software

4.5.1 Introduction

The surface temperature of a body that is not in thermal equilibrium with its surroundings changes in time. If the body is warmer than its environment, it cools as heat "flows" from the body into the environment. The simplest model of this phenomenon, *Newton's law of cooling*, holds that the rate at which the surface temperature changes in time is proportional to the difference between the ambient and surface temperatures. If the surface temperature at time t is $u(t)$, and the ambient temperature is a constant A, then by Newton's law of cooling

$$\frac{du}{dt} = \alpha(A - u),$$

where α, the heat transfer coefficient, is a positive constant. This is a classical exponential decay model, and the *direct* problem of determining the surface temperature has the unique solution

$$u(t) = A + (u(0) - A)e^{-\alpha t}.$$

This solution depends on three parameters: the ambient temperature A, the initial temperature $u(0)$, and the heat transfer coefficient α. Of course, observation of the surface temperature at appropriate times allows the solution of the *inverse* problem of determining the parameters A, $u(0)$, and α (see Exercises 2–3).

Newton's law of cooling is purely a *surface* principle; it involves a *boundary* condition, and it leads to a surface temperature that depends only on time. In extended bodies, interior temperatures typically vary not only in time but also from place to place. For example, the handle of a skillet is usually a bit cooler than the pan. The physical principles that govern internal temperatures were first explained by Joseph Fourier (1768–1830) at the beginning of the nineteenth century. Fourier's analysis hinged on the relationship between heat and temperature, and on the principle of conservation of energy.

Heat is a form of energy: The heat content of a body is a measure of the *total* kinetic energy of the molecules of the body. Temperature, as gauged by a test body (a *thermometer*), is related to the *average* kinetic energy of the molecules of a body. The heat content of a body depends not only on its temperature, but also on its mass—a 5-kilogram ball of iron at a given temperature has five times the thermal energy of a 1-kilogram ball of iron at the same temperature.

Heat is also related to the specific type of material. A 1-kilogram ball of cotton at a given temperature has less thermal energy than a 1-kilogram ball of lead at the same temperature. These ideas are bound together by the relationship

$$Q = cmu,$$

where u is the (uniform) temperature of a body, m is its mass, c is a material-dependent parameter called the specific heat of the substance, and Q is the heat content. Typical units are calories for Q, degrees Celcius for u, grams for m, and hence calories per gram-degree for c.

We limit our discussion of internal temperatures to a body with the simplest geometry—a bar of unit length and unit cross-sectional area, which we imagine to extend along the unit interval of the x-axis. Suppose the mass density and specific heat of the material of which the bar is made are ρ and c, respectively. We assume that the lateral surface of the bar is insulated so that the spatial dependence of the temperature is a function of the single variable x. The temperature of the point of the bar at position $x \in [0, 1]$ and at time $t \geq 0$ is then a function $u(x, t)$. Consider a thin slice of the bar extending over the interval $[x, x + \Delta x]$. The heat content of this slice is then about

$$c\rho u \Delta x,$$

and the rate of change of this thermal energy with respect to time is approximately

$$\frac{\partial(c\rho u)}{\partial t}\Delta x.$$

Respect for the principle of conservation of energy demands that this quantity equal the net rate of flow of thermal energy into the slice, plus the rate, if any, at which heat is produced within the slice. If the rate at which heat is produced (at position x and time t) per unit volume is denoted by $f = f(x, t)$, then the rate at which heat is produced within the slice is approximately $f\Delta x$ (remember that we assume a unit cross-sectional area).

Heat may flow into (or out of) the slice only through the left face at x or the right face at $x + \Delta x$. Fourier's law, the final ingredient in the model, states that the rate of flow of heat through a face is proportional to the negative *temperature gradient*, $-\partial u/\partial x$, at the face (the reason for the negative sign is that heat flows from hot to cold). The net flow of heat across the surface into the slice $[x, x + \Delta x]$ is therefore

$$k\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - k\frac{\partial u}{\partial x}\bigg|_{x},$$

where the proportionality constant, k, is called the thermal conductivity. Adding this to the rate at which thermal energy is generated internally, we find that the net rate of change of thermal energy within the slice is approximately

$$k\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - k\frac{\partial u}{\partial x}\bigg|_{x} + f\Delta x.$$

By the conservation of energy principle, this should match the rate calculated previously, that is,

$$\frac{\partial(c\rho u)}{\partial t}\Delta x \approx k\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - k\frac{\partial u}{\partial x}\bigg|_{x} + f\Delta x.$$

The precise model results when the interval $[x, x+\Delta x]$ is shrunk to the point x:

$$\frac{\partial(c\rho u)}{\partial t} = \lim_{\Delta x\to 0}\frac{k\frac{\partial u}{\partial x}\big|_{x+\Delta x} - k\frac{\partial u}{\partial x}\big|_{x}}{\Delta x} + f$$

or

$$\frac{\partial(c\rho u)}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) + f.$$

This is Fourier's celebrated *heat equation.*

The *direct* problem for the heat equation consists of finding the temperature $u(x,t)$ for all positions $x \in (0,1)$ and times $t > 0$, given certain boundary conditions, say the temperatures of the endpoints $u(0,t)$ and $u(1,t)$, an initial temperature distribution $u(x,0)$, and values of the parameters c, ρ, and k. These parameters are, in general, functions of space, time, and temperature.

In the Activities, we treat some relatively simple *inverse* problems involving the identification and estimation of "distributed" parameters in the heat model. Specifically, we consider the problem of determining the time-dependent parameter $a(t)$ in the problem

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \sin \pi x$$

from observations of the temperature history $h(t) = u(.5, t)$ of the midpoint of the bar.

We also propose a method for estimating the function $b(x)$ in the problem

$$b(x)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = g(x)$$

from observations of u.

Finally, we study some aspects of identifying the distributed parameter $k(x)$ in the *steady state* (i.e., $\partial u/\partial t = 0$ for all x) heat distribution problem

$$\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = -f(x), \quad 0 < x < 1.$$

4.5.2 Activities

1. Exercise Suppose u is the surface temperature of a body that cools, according to Newton's law, in an environment with constant temperature $A < u(0)$. Show that $u(t)$ is a strictly decreasing function whose graph is concave-up and has $u = A$ as a horizontal asymptote.

2. Exercise Measured surface temperatures of a body that cools according to Newton's law are given at various times in the following table:

t(min.)	u(°F)
5	72
10	62
15	54

Find the ambient temperature, the initial surface temperature, and the heat transfer coefficient.

3. Problem Show that observations of the surface temperature u of a body that cools according to Newton's law at three times $t_1 < t_2 < t_3$ uniquely determine the parameters $A, u(0)$, and α.

4. Problem Show that if a body cools according to Newton's law, then for any sequence of times $t_1 < t_2 < t_3 < \cdots$ that forms an arithmetic progession, the sequence of temperature differences $A - u(t_k)$ forms a geometric progression.

5. Exercise Suppose $a(t)$ is a positive continuous function for $t \geq 0$. Give physical interpretations for the following conditions on a function $u = u(x,t)$: