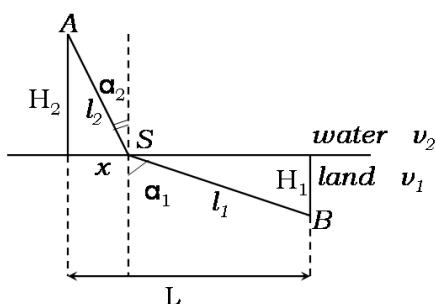


1 Elementary theory of rainbows and related phenomena

1.1 Preliminaries: the Snell Law of refraction (circa 1620)



Consider a classic Calculus I problem:

To what point S on the shore should the person swim in order to get from point A to point B in the least amount of time?

Indices ‘2’ and ‘1’ refer to water and land, respectively. The calculations below list quantities with index ‘2’ first because the person begins their motion in water and then continues on land.

Solution: Using the notations of the above figure, we have:

$$t_{\text{total}}(x) = t_{\text{water}}(x) + t_{\text{land}}(x) = \frac{l_2}{v_2} + \frac{l_1}{v_1} = \min \quad \Rightarrow$$

$$\frac{d}{dx} \left(\frac{\sqrt{x^2 + H_2^2}}{v_2} + \frac{\sqrt{(L-x)^2 + H_1^2}}{v_1} \right) = 0.$$

Straightforward calculations (verify) yield:

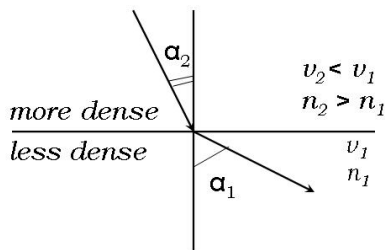
$$\frac{\sin \alpha_2}{v_2} - \frac{\sin \alpha_1}{v_1} = 0.$$

Verifying that this is indeed the minimum of $t_{\text{total}}(x)$ by Calculus tools such as the first- or second-derivative tests is, in principle, possible, but quite cumbersome. Indeed, one would need to solve for the x -value corresponding to the equation immediately above; then imagine computing $t''_{\text{total}}(x)$ and substituting in it that value of x . This would be quite difficult. A much easier way would be to simply plot $t_{\text{total}}(x)$ (e.g., using *Mathematica*).

We will now use the *Fermat’s Principle*: Light travels so as to minimize the travel time.

Then, finding the path of a refracted ray of light is mathematically equivalent to the problem above. Thus, when light refracts, it satisfies the *Snell Law*:

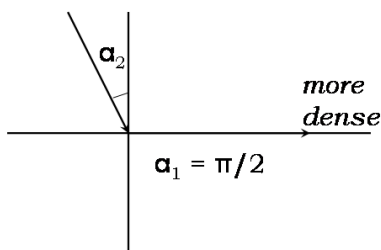
$$\frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2} \tag{1.1}$$



Since the light speed v in a medium is related to the light speed c in vacuum as $v = c/n$, where n is the medium’s refractive index, then the above equation can be rewritten as

$$\boxed{n_1 \sin \alpha_1 = n_2 \sin \alpha_2} . \tag{1.2}$$

Equation (1.2) is an alternative form of the Snell Law.



Side note: If light falls from an optically more dense medium into optically less dense one (e.g., from water into air, with $n_{\text{water}} > n_{\text{air}}$), then *total internal reflection* will occur when $\alpha_2 > (\alpha_2)_{\text{crit}}$. Here $(\alpha_2)_{\text{crit}}$ is calculated by setting $\alpha_1 = \pi/2$ (see the figure on the left). Then from (1.2) we have (verify):

$$\sin(\alpha_2)_{\text{cr}} = \frac{n_1}{n_2}.$$

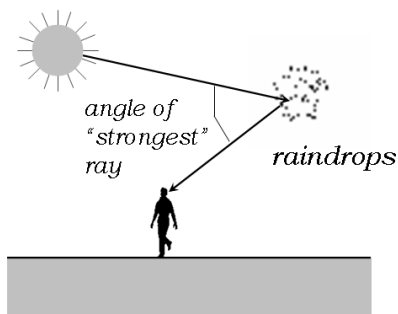
1.2 Location of the primary rainbow

1.2.1 Brief history

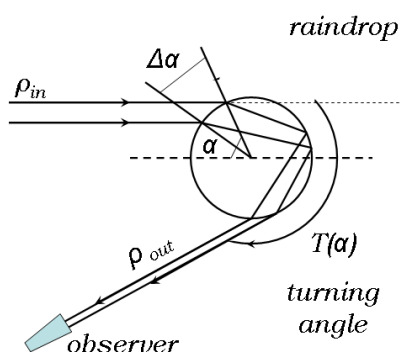
- Aristotle and other ancient philosophers:
Rainbow occurs due to refraction and reflection of sunlight in raindrops; Empirical finding of locations of primary and secondary rainbows relative to the Sun and observer.
- R. Descartes (1637):
Explained *why* rainbow is formed and *how* to find its location.
- I. Newton (1666):
Colors in rainbow (prism experiment).
- E. Halley (around same time):
Location of tertiary rainbow.
- George Airy (1838):
Explained why transition from a rainbow to the sky is not abrupt.

See the website <http://www.atoptics.co.uk> and the article by J. Adam (posted on the course website) for references on mathematical physics of rainbows.

1.2.2 Basic idea

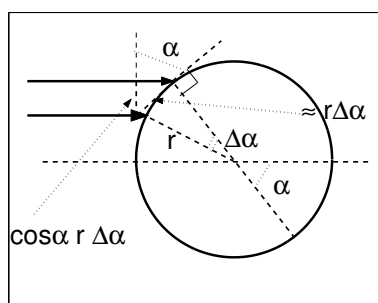


The key idea, expressed by Descartes, is that rainbows are formed due to refraction and reflection of sunlight by raindrops. More specifically, this combination of refraction and reflection occurs stronger at certain angles than it does at other angles. We will now explain why this is so.



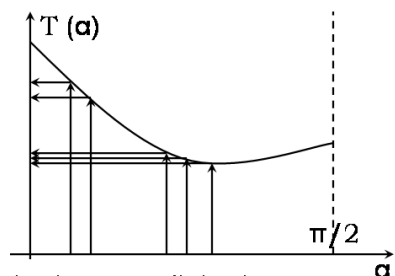
Let an incoming ray hit a raindrop at an angle α shown on the left. The ray undergoes a refraction, internal reflection, and another refraction, and as a result is turned by an angle $T(\alpha)$. So, incoming rays incident on the droplet at different angles α get turned by different angles T . All rays coming into the droplet from a narrow angular sector $\Delta\alpha$, as shown on the left, will come out of the droplet in a narrow angular sector $\Delta T(\alpha) \equiv \Delta T$. The more rays “get packed” into this $|\Delta T|$, the brighter is the image seen by the observer.

The number of rays entering the raindrop from the angular sector $\Delta\alpha$ is: $N_{in} = \rho_{in} \cdot r \Delta\alpha \cdot \cos \alpha$ (see the figure below, where the last two factors are clarified with dotted arrows). Here ρ_{in} is the density of rays coming at the droplet from one direction (in the figure, this direction is that of the horizontal). Factor $r \cdot \Delta\alpha$ is the length of the arc bounded by the two rays entering the droplet; it approximates the length of the hypotenuse of the small right triangle seen there.



The number of rays coming out of the droplet in the angular sector ΔT is, similarly: $N_{out} = \rho_{out} \cdot r \Delta T \cdot \cos \alpha$, where we have used the fact that the rays come out at the same angle α to the normal radius at which they came in: see the figure at the end of this page. Setting $N_{out} = N_{in}$, as explained above, one finds:

$$\rho_{out} = \frac{\rho_{in} \Delta\alpha}{|\Delta T|} \sim \frac{\Delta\alpha}{\Delta T}.$$



distinct input rays produce distinct output rays

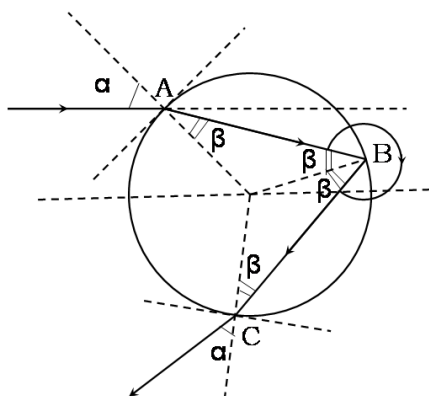
distinct input rays produce almost coinciding output rays

The refraction is the strongest into those angles where $\rho_{out}(\alpha)$ is maximum. This occurs where $|\Delta T/\Delta\alpha| = \min$. Now, $\Delta T \approx T'(\alpha)\Delta\alpha$, and therefore we look for such α where

$$T'(\alpha) = 0,$$

which is the condition for a critical point, which may be a local minimum or maximum. We will now show that $T(\alpha)$ looks as shown on the left.

1.2.3 Calculation of the rainbow location



The turning angle $T(\alpha)$ accumulates from the turns at points A,B,C as shown on the left.

Point	Turning angle
A	$\alpha - \beta$
B	$180^\circ - 2\beta$
C	$\alpha - \beta$

Thus, the total turning angle is:

$$T(\alpha) = 2(\alpha - \beta) + 180^\circ - 2\beta = 180^\circ + 2\alpha - 4 \arcsin\left(\frac{\sin \alpha}{n}\right), \quad (1.3)$$

where we have used the Snell Law (1.2) to relate angles β and α :

$$1 \cdot \sin \alpha = n \cdot \sin \beta$$

(n is the refractive index of water, while the refractive index of air is assumed to equal 1). Now,

$$T'(\alpha) = 2 - 4 \frac{d\beta}{d\alpha}.$$

To find $d\beta/d\alpha$, one can either differentiate $\beta(\alpha)$ and use the Chain Rule, or differentiate the Snell Law and use implicit differentiation. Either way, one obtains (verify):

$$T'(\alpha) = 2 - 4 \frac{\cos \alpha}{n \cos \beta}. \quad (1.4)$$

Next, we need to express $\cos \beta$ in terms of α . To that end,

$$\sin \beta = \frac{\sin \alpha}{n} \Rightarrow 1 - \sin^2 \beta = 1 - \left(\frac{\sin \alpha}{n}\right)^2 \Rightarrow \cos^2 \beta = 1 - \left(\frac{\sin \alpha}{n}\right)^2,$$

and so

$$\cos \beta = \frac{\sqrt{n^2 - \sin^2 \alpha}}{n}. \quad (1.5)$$

Substituting (1.5) into (1.4) with $T' = 0$, we obtain (verify):

$$2 = 4 \frac{\cos \alpha_0}{\sqrt{n^2 - \sin^2 \alpha_0}},$$

where α_0 is the angle where $T'(\alpha_0) = 0$. Using elementary algebra and trigonometry, one finds α_0 from the above equation as (verify):

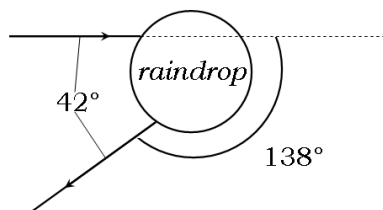
$$\cos^2 \alpha_0 = \frac{n^2 - 1}{3}, \quad (1.6a)$$

or, equivalently, as

$$\sin^2 \alpha_0 = \frac{4 - n^2}{3}. \quad (1.6b)$$

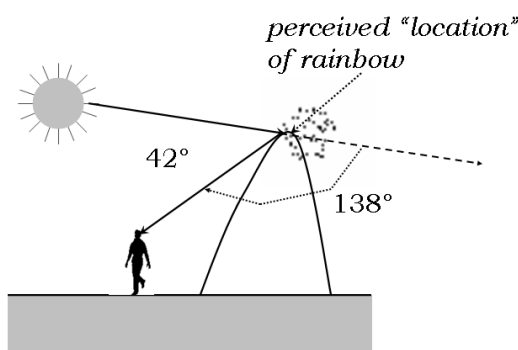
Substitution of this into (1.3) yields:

$$T(\alpha_0) = 180^\circ + 2 \arccos \sqrt{\frac{n^2 - 1}{3}} - 4 \arcsin \sqrt{\frac{4 - n^2}{3n^2}}. \quad (1.7)$$



For $n = 1.33$ (the refractive index of water), one has:

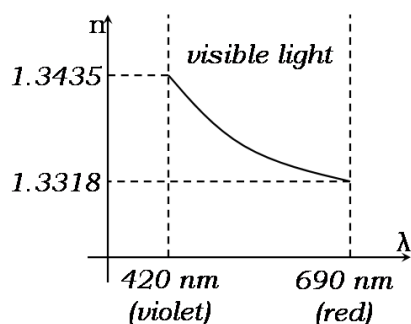
$$\begin{aligned} \alpha_0 &\simeq 59.4^\circ, \\ T_0 &\simeq 138^\circ, \\ 180^\circ - T_0 &\simeq 42^\circ. \end{aligned}$$



Thus, the first figure of Section 1.2 can be re-drawn in more detail, as shown on the left.

Note that all rays emanating from the sun are parallel because the Sun is very far from the Earth. Then the perceived location of the rainbow in the sky is also illustrated by this figure. In particular, one can conclude that the higher the sun is over the horizon, the lower the rainbow arc will appear to the observer.

1.3 Rainbow colors



The origin of colors in a rainbow was explained by Newton about 30 years after Descartes explained the mechanism for forming rainbows. When experimenting with light transmission through prisms, Newton discovered that the refractive index n depends on the light wavelength λ : $n = n(\lambda)$. This phenomenon is called *the dispersion of light* and it manifests itself in that different colors get refracted differently, i.e. are turned by a raindrop into different angles.

Brute force approach:

Find $\alpha_0(\lambda)$ for each $n(\lambda)$ from (1.6), then find $T(\alpha_0(\lambda))$ from (1.7).

Mathematically literate approach:

Use differentials (local linear approximation)! Here, we can proceed along two different venues.

Calculus I venue: Using Eqs. (1.6) and (1.7), we can compute

$$T(\alpha_0(\lambda_2)) - T(\alpha_0(\lambda_1)) \approx \frac{dT(\alpha_0(\lambda))}{d\lambda} \Delta\lambda,$$

where λ is any value from the narrow interval $[\lambda_1, \lambda_2]$. Let us denote $T(\alpha_0(\lambda)) = T_0(\lambda)$. We can find this function from (1.7) by substituting there $n = n(\lambda)$. Then we can find (conveniently using *Mathematica*) the quantity

$$\frac{dT_0}{d\lambda} = \frac{dT_0}{dn} \frac{dn}{d\lambda}, \quad \text{and hence} \quad \Delta T_0 \approx \frac{dT_0}{dn} \frac{dn}{d\lambda} \Delta\lambda,$$

where $\Delta T_0 = T_0(\lambda_2) - T_0(\lambda_1)$.

Calculus III venue is simpler. First, let

$$T(\alpha, n) = 180^\circ + 2\alpha - 4 \arcsin\left(\frac{\sin \alpha}{n}\right).$$

Then computing the differential of this function of two variables, we obtain:

$$\Delta T \approx \frac{\partial T}{\partial \alpha} \Delta\alpha + \frac{\partial T}{\partial n} \Delta n.$$

However, for the rainbow ray, i.e., for $\alpha = \alpha_0$, $\partial T / \partial \alpha|_{\alpha=\alpha_0} = 0$. Then:

$$\begin{aligned} \Delta T_0 \equiv \Delta T|_{\alpha=\alpha_0} &\approx \left. \frac{\partial T}{\partial n} \right|_{\alpha=\alpha_0} \Delta n = -4 \left. \frac{\partial}{\partial n} \arcsin \left(\frac{\sin \alpha}{n} \right) \right|_{\alpha=\alpha_0} \Delta n \\ &= -4 \left(-\frac{\sin \alpha_0}{n^2} \right) \cdot \frac{1}{\sqrt{1 - \left(\frac{\sin \alpha_0}{n} \right)^2}} \Delta n = \frac{4 \sin \alpha_0}{n \sqrt{n^2 - \sin^2 \alpha_0}} \Delta n. \end{aligned}$$

Substitute here $\sin^2 \alpha_0 = (4 - n^2)/3$ from (1.6b) to obtain (verify):

$$\left. \frac{\partial T}{\partial n} \right|_{\alpha=\alpha_0} = \frac{2}{n} \sqrt{\frac{4 - n^2}{n^2 - 1}}.$$

Then

$$\Delta T_0 \simeq \frac{2}{n} \sqrt{\frac{4 - n^2}{n^2 - 1}} \Delta n. \tag{1.8}$$

Using now $n \simeq 1.335$ and the values for $n_{\text{red}}, n_{\text{violet}}$ shown in the figure at the beginning of Section 1.3, we obtain a numeric estimate for the width of the primary rainbow:

$$T_0(\lambda_{\text{red}}) - T_0(\lambda_{\text{violet}}) \simeq -1.7^\circ; \quad \Rightarrow \quad T_{\text{red}} < T_{\text{violet}}.$$



The two figures above illustrate the following. The red ray coming from a given drop (“drop A”) goes lower than the violet ray coming from the same drop. If the violet ray from drop A enters observer’s eye, then the red ray from the same drop hits below the eye. Equivalently, the red ray must come from a drop that is *higher* than drop A to enter observer’s eye. This is why red color is seen above the violet color in a primary rainbow.

1.4 Secondary rainbow

Below we show ray paths for the primary and secondary rainbows.

Primary:



Secondary:



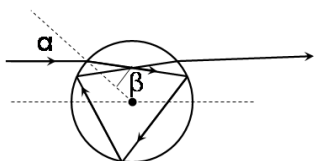
For the secondary rainbow, you will show at home that:

$$T(\alpha) = 360^\circ + 2\alpha - 6\beta \quad \text{and} \quad \cos^2 \alpha_0 = \frac{n^2 - 1}{8}. \quad (1.9)$$

You will also compute $T(\alpha_0)$, find the angular width of the secondary rainbow, and determine which color is at its top. Furthermore, if T for the secondary rainbow is interpreted in the same sense of rotation as for the primary rainbow, one can show that the secondary rainbow occurs at the local maximum of $T(\alpha)$, as shown in the figure for Sec. 1.6. You will be asked to verify this in a bonus problem.

1.5 Tertiary rainbow

The tertiary rainbow is formed by rays that have experienced *three* reflections inside the raindrop, as shown below.

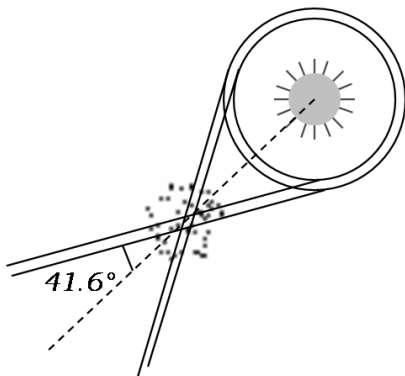


One can show that:

$$T(\alpha) = 2(\alpha - \beta) + 3(180^\circ - 2\beta),$$

$$\cos^2 \alpha_0 = \frac{n^2 - 1}{15},$$

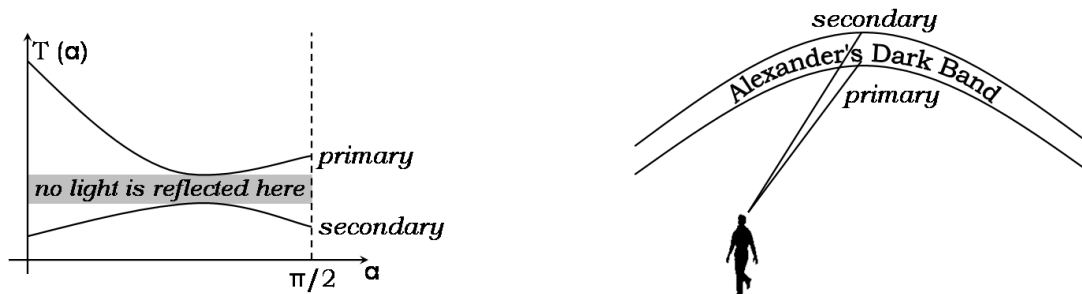
$$T(\alpha_0) \simeq 318.4^\circ.$$



Tertiary rainbows are not observed because:

- After three reflections, much light is lost, so a tertiary rainbow is faint.
- The sky is very bright near the sun.

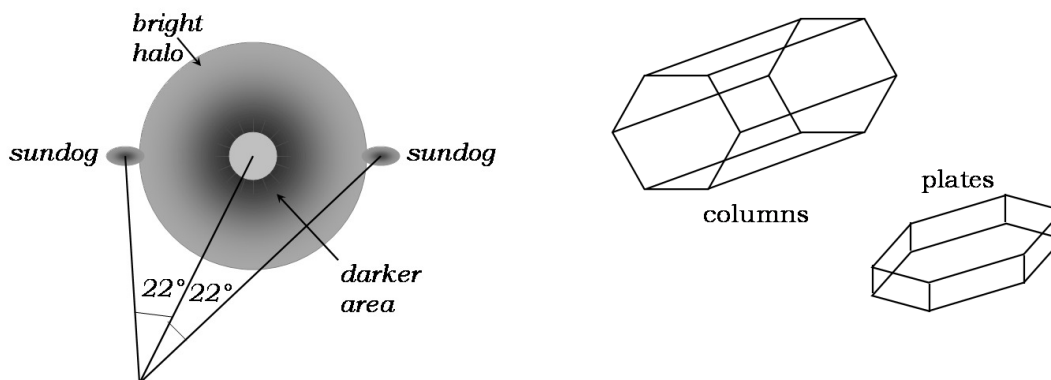
1.6 Alexander's dark band



In view of what we said in Sec. 1.4 about $T(\alpha)$ for the primary and secondary rainbows (the left figure above), there will be an angular band in the direction of the rain cloud from which no reflected and refracted light will get to the observer. In the complete and more complicated theory of rainbows than that presented above, one shows that there are no sharp boundaries between lit and unlit regions in the sky. However, the aforementioned band will appear darker than the rest of the sky; the a schematic above on the right. This band was described around 200 A.D. by a Greek philosopher Alexander, after whom it was subsequently named.

1.7 Halos

Halos occur because of refraction (not reflection!) inside certain ice crystals, which are often present in the high atmosphere even in hot climates. The most common, 22-degree, halo occurs because of the refraction in randomly oriented hexagonal columns of ice, shown below.



Finding parameters of this kind of halo follows the ideas that we have used earlier for rainbows. Namely, the turning angle $T(\alpha)$ has a minimum where $T(\alpha) \approx 22^\circ$. Then, no light is refracted inside the 22° cone (the light does get there due to other mechanisms, though), thus forming a dark area inside a bright rim. As before, the rim occurs where the density of the refracted rays is maximum, as shown below on the right. However, certain fine details of halo formation are still not fully understood.

