2 Kepler's Laws

2.1 Some history

2.1.1 End of 16th century

Tycho Brahe (pronounced [taıkou bra:hi:]), a wealthy Danish nobleman, had a passion for observation of stars. He was the director of the Prague Observatory. Since he was very rich, he could afford to construct gigantic observational instruments, with which he collected a great number of new data with precision unknown before him. (Note: a telescope was invented in 1610, nine years after Brahe's death.)

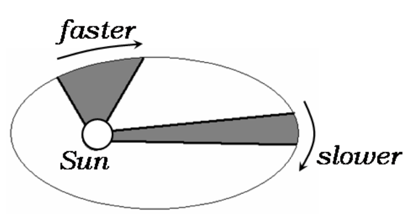
2.1.2 Beginning of 17th century

Johannes Kepler, originally from Germany, served as an assistant to Tycho Brahe at the Prague Observatory. He succeeded Brahe as the director and inherited a vast collection of observational data.

Kepler's initial ambitious goal was to describe precisely the shape of the orbit of Mars. First, he computed its period of revolution (about the Sun): $T_{\text{Mars}} = 687$(Earth)days $\simeq 1.88$ Earth years. Then, knowing the distance from the Earth to the Sun and using other astronomical data, Kepler, after many unsuccessful attempts, determined that the Mars orbit is an ellipse and the Sun is one of its foci. Moreover, he found that Mars' motion was not uniform: the farther it is from the Sun, the slower it moves. These discoveries were heretical to Kepler because they indicated that the Universe was not perfectly symmetric, something Kepler could not believe for a long time. Indeed, why is the Sun in one foci and not in the other? Yet, the data were too compelling for Kepler not to accept his new model of the elliptical motion. He then extended his model to other planets. By 1609, he formulated his first 2 laws:

(i) Each planet moves on an ellipse with the Sun at one focus.

(ii)



For each planet, the line from the Sun to the planet sweeps out equal areas in equal times.

It took Kepler 10 more years to formulate his 3rd law:

(iii) The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit:

$$\frac{T^2}{a^3} = \text{ const (same for all planets).}$$

How could Kepler find the exponents 2 and 3? He used recently invented logarithms! Namely, suppose that two given lists of data, x and y are related by

$$y = k \cdot x^n \,.$$

How can we find k and n? Answer: Take the logarithm of the above equation to obtain

$$\log y = \log k + n \log x \,.$$

This equation defines a *linear relationship* (a straight line) between $\log y$ and $\log x$; its slope is n and the intercept is $\log k$.

At home you will be asked to verify Kepler's Third law given the data for the planets from Mercury to Pluto.

2.1.3 End of 17th century

In 1679, Robert Hooke wrote a letter to Isaac Newton in which he, among other things, wrote about his two new hypotheses:

- the gravitational force of attraction between two bodies falls off as $1/r^2$, where r is the distance between them;
- bodies in such a field move along ellipse-looking orbits. (Hooke's scientific ethics did not allow him to say "ellipses" because he did not have any rigorous proof that they were not some other kind of ovals. In fact, some astronomers at that time believed that those orbits could be different, i.e., non-elliptical, curves. For example, Giovanni Cassini believed that they were special ovals, which were later named after him; you can read about Cassini ovals online.)

Science historians differ in their opinion whether this letter might have set Newton on a path of *deriving* the inverse-square law for the gravitational force from the three laws of Kepler. However, Newton himself never acknowledged any significance of Hooke's letter in his (i.e., Newton's) research on gravity. His argument to their common friend Edmund Halley about it was that all Hooke had done was to get an intuitive idea, while it was Newton who *rigorously proved* it. Relations between Hooke and Newton first became tense in 1672, when Hooke, a famous scientist at that time, criticized the optical theory of then a much less known Newton. The 1679 correspondence between the two men has led to their second falling-out. Finally, the third and last argument about priority of scientific discoveries, that re-fueled their acrimony towards each other, occurred in late 1680s after Newton published his seminal work "*Principia Mathematica*", in which he derived the Universal Gravitation law and Kepler's laws. Newton's hatred of Hooke was so strong that, when he became the President of the Royal Society already after Hooke had died, he ordered to destroy not only all works and apparatuses of Hooke[1], but even all Hooke's portraits. You may find further details about the entangled relationship of these two great scientists in their biographies, e.g., online.

[1] Hooke used to be the Curator of the Royal Society, where his responsibilities included regular demonstrations of new experiments at the Society's meetings

2.2 Newton's law of Universal Gravitation

Although Newton knew Calculus (which he invented[2] around 1670), he did not use it in his seminal book "Principia Mathematica". Instead, he used elementary geometrical methods. An approach similar to Newton's is described in an article by Hall & Higson, posted online. Here we will use the modern approach based on Calculus, but will also try to reconstruct the main steps that Newton had to follow.

In this section, we describe the two steps that Newton had to accomplish to derive his Universal Law of Gravitation (assuming that he knew his Second law, $m\vec{a} = \vec{F}$) from Kepler's laws. Namely, we will use Kepler's laws to show that:

1. the gravitational force is central (see below);

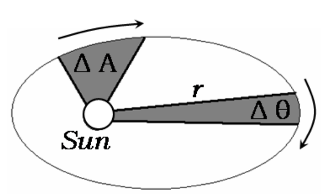
2. its magnitude follows the $1/r^2$ dependence.

2.2.1 Showing that the gravitational force is central

A force is called central its vectors at any point in space are directed towards, or away from, the same point, called the center. In the case of the gravitational force between a planet and the Sun, the Sun is such a center (see the figure below).

We will prove the statement in the title of this subsection by showing that $\ddot{\vec{r}} \propto \vec{r}$ and then invoking the Second Law of Newton, $\vec{F} = m\vec{a} \equiv m\ddot{\vec{r}}$. In the process, we will also need to invoke an assumption[3] that the plane where the planetary orbit lies remains constant in time (i.e., does not precess).

Before we begin, let us make one more remark about the derivation forthcoming in this section. It uses polar coordinates, which you learned in Calculus II, and their relation to Cartesian coordinates. This derivation is somewhat technical and may even appear as conceptually obscure. There exists a more conceptually straightforward derivation, based not on polar coordinates but on certain properties of the cross-product. In fact, this other derivation in the *reverse order* is presented later in Section 2.4. The reason why the less straightforward, polar-Cartesian, derivation is given here is to give you the opportunity to review the important topic of polar coordinates.



The Second Kepler's law says that the planet sweeps out the same area ΔA during an interval Δt anywhere along its orbit. This area can be computed using polar coordinates r and θ to be:

$$\Delta A \simeq \frac{1}{2} r^2 \Delta\theta \,,$$

since $r \simeq$ const when $\Delta\theta$ is small.

Let us notice a subtle issue: The reason why one could use polar coordinates (which provide an alternative description to the Cartesian coordinates x and y in *two dimensions*) is the aforementioned assumption that the plane in which the planet moves remains constant in time.

[2]This invention of Newton led to his long-term feud with another great mind of the XVII century — Gottfried Leibniz — who invented Calculus independently and around the same time.

[3]Whether or not it was stated by Kepler as an observational fact is unknown to the author of these notes.

If that plane had precessed (i.e. had moved in three dimensions), then a third coordinate would have been needed in addition to the two in-plane polar coordinates.

Continuing with our derivation and dividing the last expression by Δt yields:

$$\frac{\Delta A}{\Delta t} \simeq \frac{1}{2}r^2\frac{\Delta\theta}{\Delta t}, \qquad \text{or} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

Let us introduce a notation that will be used extensively in this course:

$$\frac{dq}{dt} = \dot{q}, \qquad \text{for any quantity } q.$$

Thus, the Second Kepler's law can be written as:

$$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \text{const} \equiv \frac{1}{2}c. \tag{2.1}$$

Now, as we will explain in a moment,

$$r^2\dot{\theta} = x\dot{y} - \dot{x}y. \tag{2.2}$$

Indeed,

$$x = r\cos\theta, \qquad y = r\sin\theta; \qquad \dot{x} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta, \qquad \dot{y} = \dot{r}\sin\theta + \dot{\theta}r\cos\theta. \tag{2.3}$$

Substituting these expressions into the r.h.s. of (2.2), we obtain its l.h.s. (verify). Since this l.h.s. = const (see (2.1)), then

$$\frac{d}{dt}(x\dot{y} - \dot{x}y) = 0 \qquad \Rightarrow \qquad x\ddot{y} - \ddot{x}y = 0 \quad \text{(verify)}. \tag{2.4}$$

Next, (2.4) can be used to show that $\vec{r}\times\ddot{\vec{r}} = \vec{0}$. Indeed, if $\vec{r} = \langle x, y, 0\rangle$, $\ddot{\vec{r}} = \langle \ddot{x}, \ddot{y}, 0\rangle$, then $\vec{r}\times\ddot{\vec{r}} = \vec{i}\cdot 0 + \vec{j}\cdot 0 + \vec{k}(x\ddot{y} - \ddot{x}y) = \vec{0}$ (verify), which means $\vec{r}\parallel\ddot{\vec{r}}$, or equivalently,

$$\ddot{\vec{r}} = f(r,\theta)\vec{r} \tag{2.5a}$$

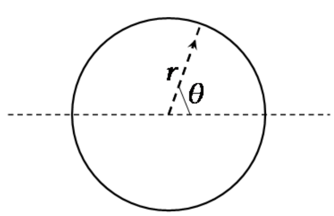
for some scalar function $f(r,\theta)$. Finally,

$$m\vec{a} = \vec{F} \qquad \Rightarrow \qquad m\ddot{\vec{r}} = \vec{F} \qquad \Rightarrow \qquad \vec{F} = mf(r,\theta)\vec{r}, \tag{2.5b}$$

i.e. the force is collinear with $\vec{r}$ and hence is central.

2.2.2 Showing that $f(r,\theta) = \text{const}/r^2$

The planets' orbits are, in fact, nearly circular (and this was known to Kepler and Newton). Let us assume for now that the orbits are exactly circular. Then, using the polar equation of a circle, we have:



$$\vec{r} = \langle r\cos\theta, r\sin\theta\rangle$$

$$r = \text{const}, \qquad \theta = \frac{2\pi t}{T},$$

where T is the period of the planet's revolution about the Sun. By differentiating the first line above and using the second line, one finds (verify):

$$\ddot{\vec{r}} = -\left(\frac{2\pi}{T}\right)^2\vec{r}.$$

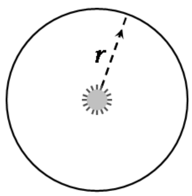
Comparing this with Eq. (2.5a), we have:

$$f(r,\theta) = -\frac{4\pi^2}{T^2}\,.$$

But by Kepler's Third law (applied to circular orbits), $T^2 = \text{const}\cdot r^3$. Hence $f(r,\theta) \sim 1/r^3$ and therefore

$$\vec{F} = -\frac{\text{const}}{r^3}\vec{r}, \qquad \text{or} \qquad \|\vec{F}\| \sim \frac{1}{r^2}\,. \tag{2.6}$$

Side note: Kepler also thought of the inverse-square law for the gravitation, but rejected the idea. His reasoning was approximately the following.



Kepler thought of the gravitational field emanated by a body as being similar to light. The intensity of light away from its source falls off as $1/r^2$:

$$\frac{\text{source strength}}{\text{surface area}} = \frac{\text{const}}{4\pi r^2}\,.$$

So, it then would be natural that the intensity of the emanated gravitational field also falls off as $1/r^2$.

However, if this model were correct, then it would contradict to what happens during the solar eclipse. In this event, the Moon blocks the Sun's light (and, as Kepler thought, gravitation) from reaching the Earth, and hence during solar eclipses, the motion of the Earth should experience large perturbations. Since this was never observed, Kepler rejected the inverse-square law for gravitation.

The reason this theory (and its rejection by Kepler) became known to us is that Kepler in his publications did not "try to cover his traces" (unlike most scientists). He wrote about his conjectures, failures, successes, errors, insights, etc. with great frankness. The downside of this style is that when many competing ideas are presented in a paper, most readers are lost. So it took Newton's genius to separate the wheat from the chaff and discern the importance of Kepler's three laws.

2.3 Auxiliary formulae from vector Calculus

We will now proceed in the reverse direction: We will use Newton's Law of Universal Gravitation to derive the three Laws of Kepler. For that, we need to remind or establish four auxiliary facts.

Fact 1 For any three vectors $\vec{a}, \vec{b}, \vec{c}$:

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - \left(\vec{b}\cdot\vec{c}\right)\vec{a}\,. \tag{2.7}$$

This formula can be established by direct calculation. However, below we will illustrate a more mathematically literate method. Its idea is to expand arbitrary vectors in the basis of unit coordinate vectors and then verify that the desired identity holds for those basis vectors. While the verification of the identity for one triple of basis vectors is easier to do than for arbitrary vectors, the *total* amount of work needed to follow the basis-vector approach may be actually larger than the amount of work involved in direct calculation (see below). The point of the basis-vector approach is then *not* in making one's life easier, but in elucidating the fact that the formula for the arbitrary vectors holds because it holds for the basis vectors, which can be viewed as building blocks from which an arbitrary vector is made.

Let

$$\vec{a} = \sum_{l=1}^{3} a_l \vec{e_l}\,, \qquad \vec{b} = \sum_{m=1}^{3} b_m \vec{e_m}\,, \qquad \vec{c} = \sum_{n=1}^{3} c_n \vec{e_n}\,, \qquad \{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \{\vec{i}, \vec{j}, \vec{k}\}\,,$$

where a_l are the usual Cartesian coordinates of $\vec{a}$, etc. Then the l.h.s of (2.7) can be rewritten as:

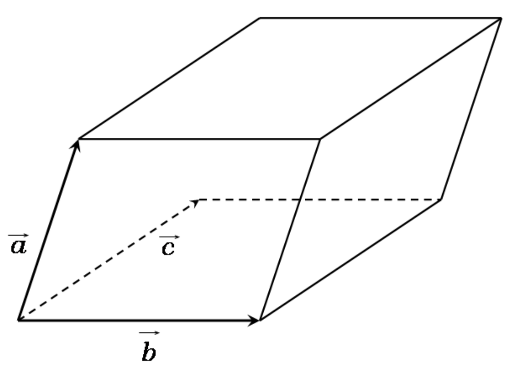
$$\sum_{n=1}^{3}\sum_{m=1}^{3}\sum_{l=1}^{3} a_l b_m c_n \cdot (\vec{e_l} \times \vec{e_m}) \times \vec{e_n}\,.$$

Therefore, since a_l, b_m, c_n can be arbitrary, it suffices to verify that (2.7) holds for the unit coordinate vectors $\{\vec{i}, \vec{j}, \vec{k}\}$ in different combinations. Now, out of 27 such combinations, 9 with $l = m$ are immediately seen to be zeros. Another 6, where all of l, m, n are different, are also zero, due to the property $\vec{i} \times \vec{j} = \vec{k}$ and its permutations. Thus, "only" 12 combinations, with $l \neq m$ and $n = l$ or m are to be considered. But those are verified easily, e.g.:

$$\text{l.h.s.} = (\vec{i} \times \vec{j}) \times \vec{i} = \vec{k} \times \vec{i} = \vec{j};$$

$$\text{r.h.s.} = (\vec{i} \cdot \vec{i})\,\vec{j} - (\vec{j} \cdot \vec{i})\,\vec{i} = 1\,\vec{j} - 0\,\vec{i} \;=\; \text{l.h.s.}$$

Fact 2



Recall that the *triple product* is related to the volume of a parallelepiped whose sides are $\vec{a}, \vec{b}, \vec{c}$:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|\,.$$

Also,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}\,. \qquad (2.8)$$

Fact 3 Suppose $\vec{r}(t)$, and hence $\dot{\vec{r}}(t)$, are known. We can explicitly separate the length and direction of this vector:

$$\vec{r} \equiv r \cdot \left(\frac{\vec{r}}{r}\right)\,,$$

where r is the length and $(\vec{r}/r)$ is the unit vector along $\vec{r}$. Then, how can we relate $\dot{r}$ and $(\vec{r}/r)^{\cdot}$ to $\vec{r}$ and $\dot{\vec{r}}$? First, let us find $\dot{r}$:

$$r^2 = (\vec{r} \cdot \vec{r}) \qquad \Rightarrow \qquad (r^2)^{\cdot} = (\dot{\vec{r}} \cdot \vec{r}) + (\vec{r} \cdot \dot{\vec{r}}) \qquad \Rightarrow \qquad 2r\dot{r} = 2(\dot{\vec{r}} \cdot \vec{r}) \qquad \Rightarrow \qquad \dot{r} = \frac{(\dot{\vec{r}} \cdot \vec{r})}{r}\,. \qquad (2.9)$$

Note that r is related to $\vec{r}$ by the first formula in (2.9), and then $\dot{r}$ is defined in terms of $\vec{r}$ and $\dot{\vec{r}}$ by the last formula.

Next, using that last formula and the Product Rule

$$\dot{\vec{r}} = \left(r \left(\frac{\vec{r}}{r} \right) \right)^{\cdot} = \dot{r} \left(\frac{\vec{r}}{r} \right) + r \left(\frac{\vec{r}}{r} \right)^{\cdot},$$

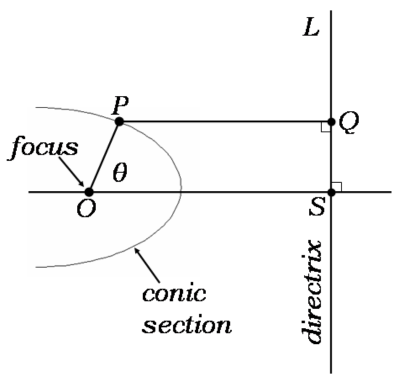
one finds (verify):

$$\left(\frac{\vec{r}}{r} \right)^{\cdot} = \frac{(\vec{r}\cdot\vec{r})\,\dot{\vec{r}} - (\dot{\vec{r}}\cdot\vec{r})\vec{r}}{r^3}.$$

Finally, using Eq. (2.7), we obtain a formula that we will use in the next section:

$$\left(\frac{\vec{r}}{r} \right)^{\cdot} = \frac{(\vec{r}\times\dot{\vec{r}})\times\vec{r}}{r^3}. \tag{2.10}$$

Fact 4 Conic sections in polar coordinates.



Definition A conic section is a planar curve whose points P satisfy the following property:

$$|OP| = e|PQ|$$

where the notations are as shown in the figure and e is a fixed number called *eccentricity*. (Note that for a *given* conic section, the locations of the focus and the directrix are not arbitrary but rather must be carefully selected.)

On p. 71 of the book "Inverse Problems" by C.W. Groetsch (MAA 1999), it is shown that the polar equation of a conic is

$$r = \frac{eD}{1 + e\cos\theta}, \tag{2.11}$$

where $D = |OS|$ is the (given) distance between the focus and the directrix.

In a homework problem you will show that Eq. (2.11) reduces to more familiar Cartesian forms of three different conic sections depending on the value of e.

With this background information, we are ready to derive the three Kepler's laws from Newton's Law of Universal Gravitation (Eq. (2.6)) and the Second law of motion, $m\vec{a} = \vec{F}$.

2.4 Derivation of Kepler's laws from Newton's laws

We begin by restating Newton's Law of Universal Gravitation, (2.6), and the Second law of motion as one differential equation:

$$\ddot{\vec{r}} = -\frac{N}{r^3}\vec{r} \tag{2.12}$$

where N is a proportionality constant. From this, we will derive the first two Kepler's laws. In fact, it is convenient to derive Kepler's Second law before the First, and so below we will state them in this order and also will slightly modify the statement of the Second law.

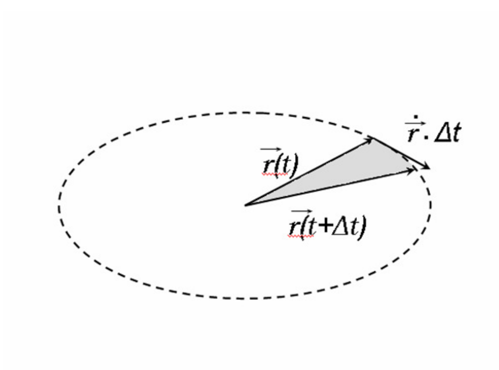
- *Kepler's Second law, slightly modified*
  A planet's trajectory around the Sun is a planar curve (i.e., lies in one plane), and the radius-vector from the Sun to the planet sweeps out equal areas in equal time intervals.

  The aforementioned modification is contained in the first clause (about the planar curve) of the previous sentence. It is actually borrowed from the First law, where one does not specify the shape of that planar curve. Then the First law specifies what that shape is:

- *Kepler's First law*
  The trajectory of each planet is an ellipse with the Sun at one of its foci.

Derivation of Kepler's Second law

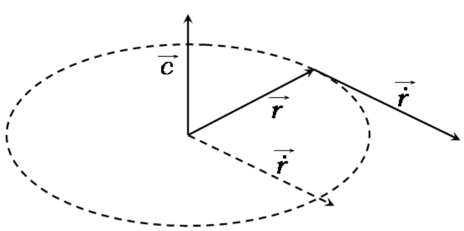
Since we want to prove a statement about the area of a sector swept by the radius-vector of a planet, we begin by establishing a formula for that area.



It (the area) is shown in the figure on the left and is seen to be approximately equal to the area of the triangle with one side $\vec r$ and the base $\dot{\vec r}$. From vector Calculus, you should recall that this area equals one half of $|\vec r \times \dot{\vec r}\Delta t|$. Thus, the area swept by the radius-vector per time Δt is proportional to $|\vec r \times \dot{\vec r}|$, and hence to demonstrate the second clause in the modified Second Kepler's law stated above, we need to show that

$$|\vec r \times \dot{\vec r}| = \text{const.} \tag{2.13}$$

Let us now use the first clause of that law to show that we need the *vector* $\vec r \times \dot{\vec r}$, not just its length, to stay constant.



Indeed, if the trajectory lies in a plane, one can define a normal vector to that plane, and the direction of this normal vector stays constant at all times. This normal vector, by definition, is perpendicular to any vector in the plane of motion and hence has the direction of $\vec r \times \dot{\vec r}$, by a property of the cross product. Hence the direction of $\vec r \times \dot{\vec r}$ must stay constant. Along with (2.13) this implies that we want to show that

$$\vec r \times \dot{\vec r} = \vec{\text{const}} \equiv \vec c\,. \tag{2.14}$$

This is shown as follows. Take the cross-product of Eq. (2.12) with $\vec r$ and use a property of the cross product to see that

$$\vec r \times \ddot{\vec r} = \vec 0\,. \tag{2.15}$$

Next, using the Product Rule for $(\vec r \times \dot{\vec r})^{\cdot}$ and substituting in one of the terms (2.15), one obtains the second equation below (verify):

$$(\vec r \times \dot{\vec r})^{\cdot} = \vec r \times \ddot{\vec r} + (\dot{\vec r} \times \dot{\vec r}) \qquad \Rightarrow \qquad (\vec r \times \dot{\vec r})^{\cdot} = \vec 0\,. \tag{2.16}$$

By integrating the last equation, one obtains (2.14).

Let us note in passing that

$$\vec{c} = \vec{r} \times \dot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = \vec{k}(x\dot{y} - \dot{x}y)\,.$$

Therefore, $|\vec{c}| = |x\dot{y} - \dot{x}y|$, which has been shown to be a constant by a different method: see (2.1) and (2.2).

Derivation of Kepler's First law

Our goal is now to show that a solution of the differential equation (2.12) is an ellipse. If our differential equation had the form $\dot{\vec{r}} = \vec{f}(t)$, one could integrate it once to get the solution:

$$\vec{r}(t) = \int_{t_0}^{t} \vec{f}(t')dt' + \vec{r_0}\,.$$

Similarly, to solve $\ddot{\vec{r}} = \vec{f}(t)$, one would integrate twice. However, in Eq. (2.12), the r.h.s. itself depends on the unknown $\vec{r}(t)$. Moreover, the solution we are looking for does not have the form $\vec{r} = \vec{r}(t)$ but, instead, has the form $r = r(\theta)$: see (2.11). So we have to devise a trick to find that solution. This trick has two steps.

Step 1: Integrate (2.12) *once to get an equation for* $\dot{\vec{r}}$

In Section 2.3 we showed that (see (2.10)) for *any* law of motion,

$$\left(\frac{\vec{r}}{r}\right)^{\cdot} = \frac{\vec{c} \times \vec{r}}{r^3}\,, \qquad (2.17a)$$

where $\vec{c}$ is defined in (2.14). The r.h.s. of Eq. (2.17a) resembles the r.h.s. of the Newton's equation (2.12) if we multiply the latter by $\vec{c}$:

$$\vec{c} \times \ddot{\vec{r}} = -\frac{N}{r^3}\,(\vec{c} \times \vec{r})\,. \qquad (2.17b)$$

Let us multiply both sides of (2.17a) by $-N$ and compare the result with (2.17b). This comparison yields:

$$-N\left(\frac{\vec{r}}{r}\right)^{\cdot} = (\vec{c} \times \dot{\vec{r}})^{\cdot}\,;$$

to obtain the r.h.s. of this equation, we have used the fact that $\vec{c} = \vec{\text{const}}$, whence $\vec{c}\times\ddot{\vec{r}} = (\vec{c}\times\dot{\vec{r}})^{\cdot}$. Integrating this once and using the identity $\vec{c} \times \dot{\vec{r}} = -\dot{\vec{r}} \times \vec{c}$, we get

$$\dot{\vec{r}} \times \vec{c} = N\left(\vec{e} + \frac{\vec{r}}{r}\right)\,, \qquad (2.18)$$

where $\vec{e}$ is the integration constant.

Let us now show that $\vec{e}$ must be perpendicular to $\vec{c}$. Recall that in order to show that any two vectors, $\vec{a}$ and $\vec{b}$, are perpendicular, one needs to show that $\vec{a} \cdot \vec{b} = 0$. Then, we want to show that $\vec{e} \cdot \vec{c} = 0$. Let us dot-multiply (2.18) by $\vec{c}$; then $\vec{e} \cdot \vec{c}$ will be one of the two terms on the r.h.s.. The other term on the r.h.s., that proportional to $\vec{r} \cdot \vec{c}$, is zero since $\vec{r} \perp \vec{c}$ by (2.14) and the property that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. By the same property, the term on the l.h.s. also vanishes: $(\dot{\vec{r}} \times \vec{c}) \cdot \vec{c} = 0$. Therefore,

$$\vec{e} \cdot \vec{c} = 0, \qquad \text{i.e., indeed,} \qquad \vec{e} \perp \vec{c}\,.$$

Then, according to (2.14) and the paragraph before it, $\vec{e}$ lies in the plane of motion.

Step 2: Eliminate $\dot{\vec{r}}$ *from* (2.18)

We still need to perform the task of eliminating $\dot{\vec{r}}$ from (2.18). To do so, we begin by taking the dot product of (2.18) with $\vec{r}$ and obtain:

$$(\vec{r} \times \dot{\vec{r}}) \cdot \vec{c} = N(\vec{r} \cdot \vec{e} + r)\,,$$

where on the l.h.s. we have used (2.8). Using the definition of $\vec{c}$ from (2.14), we rewrite the last equation as

$$c^2 = N(\vec{r} \cdot \vec{e} + r)\,. \tag{2.19}$$

Also, since $\vec{e}$ lies in the plane of motion, we can write:

$$\vec{r} \cdot \vec{e} \equiv r \cdot e \cdot \cos\theta\,,$$

where θ is shown in the figure at the end of Sec. 2.3. Finally, from the last equation and (2.19), we get (verify):

$$r = \frac{\left(\dfrac{c^2}{Ne}\right) e}{1 + e\cos\theta}\,, \tag{2.20}$$

which is the Equation (2.11) of a conic section.

Derivation of Kepler's Third law

You will do it in a homework problem following the hints provided there.

As a note about notations, recall that Kepler's Third law is:

$$T^2 \,:\, a^3 = \text{const.} \tag{2.21}$$

Then, if T is measured in (Earth) years and a is measured in Astronomical units (1 A.U. = half of the major axis of the Earth orbit, or approximately the average distance from Sun to Earth), the Third law takes on a simple form:

$$T^2 \,:\, a^3 = 1$$

(why?). For example, the Halley's comet (last seen in 1986, eccentricity 0.97) has a semi-major axis $a = 18.1$ A.U. When will it be seen again?