3 Which is faster, going up or coming down?

Suppose you throw a ball into the air. If one neglects the air resistance, then it takes the same time for the ball to reach the highest point of its trajectory and to fall down from that point to the Earth. Now, if one does take the air resistance into account, which way will it take the ball longer to go, up or down?

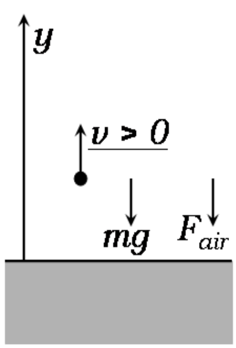
It can be shown, both rigorously and at the intuitive level, that it will take longer for the ball to fall than to rise. This holds for *any* law of air resistance, as long as the motion of the ball is strictly vertical (i.e., one-dimensional) and the air resistance depends only on the ball's velocity. For details see the article by F. Brauer posted online.

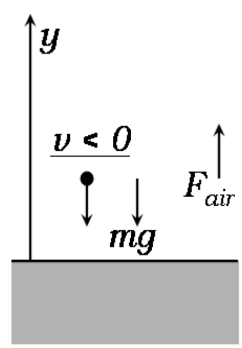
In this lecture, we will answer the above question in a much restricted setting, namely: when the air resistance is very small (compared to the gravity) and, moreover, when its magnitude is a simple linear or quadratic function of the velocity. The reason we will study this restricted setting is that we will illustrate some basic steps of the *perturbative approach* to solving algebraic and differential equations. We will also see some applications of the *Taylor series expansion* of functions.

3.1 The exact model

3.1.1 Setting up the equation

Let a ball with mass m be moving vertically (up or down), and its initial velocity (pointing up) be v_0. We assume that the only two forces acting on the ball are the gravity and the air resistance, with the latter being directed oppositely to the direction of the ball's motion.





Projecting Newton's Second Law

$$m\vec{a} = \sum \vec{F}$$

on the y-axis, we have the following equations (see the figure above).

Going up, $v > 0$:

$$m\frac{dv}{dt} = -mg - F_{\text{air}}\,; \tag{3.1a}$$

Going down: $v < 0$:

$$m\frac{dv}{dt} = -mg + F_{\text{air}}\,. \qquad (3.1b)$$

We begin by taking F_{air} to be proportional to the first power of the velocity, i.e.

$$F_{\text{air}} = D|v|. \qquad (3.2)$$

We will refer to the constant D as the 'drag coefficient'. It is known that for small bodies (e.g., a tennis ball) with not very large velocities (e.g., falling from the roof of a few-story building), (3.2) is a good model. On the other hand, for larger objects having larger velocities (e.g., a skydiver or a parachutist), the air resistance is proportional to the *square* of the velocity:

$$|F_{\text{air}}| = Dv^2. \qquad (3.3)$$

See Sections 2 and 3 of the article by L. Long and H. Weiss posted online if you want to see some physical arguments behind this.

For now, we focus on model (3.2). In more detail, it can be written as:

$$\begin{array}{lll} \underline{v>0} \quad & F_{\text{air}} &= D\cdot v\ (>0) \\ \underline{v<0} \quad & F_{\text{air}} &= D\cdot(-v)\ (>0). \end{array}$$

Substitution of this into (3.1) yields:

$$m\frac{dv}{dt} = -mg - Dv \qquad (3.4)$$

for both the upward and downward motions.

It is customary to nondimensionalize equations. However, in the main part of this Lecture we will use a slightly different approach: we will *normalize* our equation of motion (3.4) to the constant g. Strictly speaking, this is incorrect, since variables in our normalized equation will remain dimensional, and hence characterizing them as being either small or of order one (see below) will be mathematically ill-defined. Despite this drawback, our incorrect approach has a couple of merits. First, it is conceptually more straightforward than a correct approach. Second, as we will show, there exist two different ways to normalize our (seemingly simple!) Eq. (3.4); therefore, deciding which of the ways is to be chosen is a nontrivial issue. Since our focus in this Lecture is *not* on learning how to nondimensionalize equations but rather on how to solve them perturbatively, we have adopted the conceptually simpler but mathematically incorrect normalization (instead of nondimensionalization), described below. The two mathematically correct approaches that lead to nondimensionalization of (3.4) are described in Appendix A.

Let us proceed with normalizing the time variable in Eq. (3.4). We will do this in two steps. First, dividing by m yields

$$\frac{dv}{dt} = -g - \frac{D}{m}v.$$

Second, let us introduce a new variable $\tau = gt$. Then, using the Chain Rule, we have:

$$\frac{dv}{dt} = \frac{dv}{d\tau}\cdot\frac{d\tau}{dt} = \frac{dv}{d\tau}\cdot g.$$

Substituting this into the previous equation and cancelling by g, we obtain:

$$\frac{dv}{d\tau} = -1 - \frac{D}{mg}v.$$

Finally, we use the notation introduced in Lecture 2:

$$\frac{dv}{d\tau} \equiv \dot{v}\,.$$

Then the normalized form of Eq. (3.4) is:

$$\dot{v} = -1 - Kv, \tag{3.5}$$

where $K \equiv D/(mg)$. Note that due to the change of variables from t to τ, the usual equation $\dfrac{dy}{dt} = v$ now has the form

$$\dot{y} = \frac{1}{g}v. \tag{3.6}$$

The initial conditions for (3.5) and (3.6) are:

$$v(0) = v_0, \qquad y(0) = y_0. \tag{3.7}$$

In the specific case we are considering, one has $y_0 = 0$.

Before moving on, let us comment on what we have achieved with our normalization. The first term (i.e., '-1') on the r.h.s. of Eq. (3.5) basically tells us that we no longer need to be concerned with the numeric value of g, since in the normalized variables, gravity causes the velocity to change with the rate of exactly one. Thus, one can think of g as defining the 'order one' time scale of our problem:

$$\tau \sim 1.$$

This 'order one' concept, which should be only intuitively clear at this point, will be discussed in more detail in Section 3.2.

Next, we will make two comments about the second term on the r.h.s. of (3.5). First, the constant K there is the normalized drag coefficient. Since it is *not* nondimensional (Kv is, as it is Kv that is subtracted from the nondimensional '-1'), one cannot, strictly speaking, characterize K as small or large. However, we will use statements such as "K is small" and "Kv is small" synonymously. (Recall that a more mathematically rigorous treatment of the drag's magnitude is given in Appendix A.) Second, the size of Kv compared to '1' determines how large the effect of the drag is compared to that of gravity. If $Kv \gg 1$ (the symbol '$\gg$' means 'much greater'), then the drag dominates, while $Kv \ll 1$ means the opposite. In this lecture we will primarily be interested in the latter case, i.e. when the drag is small.

3.1.2 Exact solution and its limiting cases

When the drag is absent ($K = 0$ in (3.5)), only gravity acts on the ball, and we get the familiar formulae from (3.5), (3.6), and (3.7):

$$v = v_0 - \tau \tag{3.8a}$$

$$y = y_0 + \frac{1}{g}\left(v_0\tau - \frac{\tau^2}{2}\right). \tag{3.8b}$$

(Verify that they are equivalent to the form you are used to.) This is the drag-free solution of Eq. (3.5).

Now let us consider the case $K \neq 0$ (we assume $K > 0$). The differential equation (3.5) is both separable and linear. From the ODE course you may recall that these are the two main classes of (first-order) equations that can be solved, each by its own method. In the following, we will solve (3.5) as a *separable* equation, because it is this method that you will need to use when solving counterparts of (3.5) for the quadratic air resistance law in Section 3.3. Separating variables in (3.5), one obtains:

$$\frac{dv}{d\tau} = -1 - Kv$$

$$\Rightarrow \qquad \int \frac{dv}{1+Kv} = -\tau + C \quad (C = \text{const})$$

$$\Rightarrow \qquad \frac{1}{K}\ln(1+Kv) = -\tau + C$$

(verify). Using the first initial condition for v in (3.7), we find C:

$$C = \frac{1}{K}\ln(1+Kv_0).$$

Finally, we solve for v to obtain (verify):

$$v = \frac{1}{K}\left((1+Kv_0)e^{-K\tau} - 1\right). \tag{3.9}$$

Using now (3.6) and the second initial condition in (3.7), we obtain (verify):

$$y = y_0 + \frac{1}{gK}\left((1+Kv_0)\frac{1-e^{-K\tau}}{K} - \tau\right). \tag{3.10}$$

Question: We have obtained an answer. How do we know that we haven't made a mistake?
Answer: Use sanity check — verify if the limiting cases make sense.

There are two limiting cases: $K \gg 1$ (very large) and $K \ll 1$ (very small). In the former case, we expect that the ball will go up by a small amount only (why?), and in the latter case we expect that the answer is close to that given by (3.8).

$\underline{K \gg 1}$ For any fixed τ (i.e. when τ is in no way related to K), (3.10) yields:

$$y|_{K\gg 1} = y_0 + \frac{1}{gK}\left(\underbrace{\frac{1+Kv_0}{K}}_{\approx v_0}\cdot\underbrace{(1-e^{-K\tau})}_{\approx 1-0} - \tau\right) \approx y_0 + \frac{1}{gK}(v_0 - \tau),$$

i.e., indeed, the elevation of the ball is very small.

Before we consider the other limiting case, let us introduce a new notation. Suppose ε is a small number: $\varepsilon \ll 1$. Then one says that a function $f(\varepsilon)$ is $O(\varepsilon)$ if

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\varepsilon} = \text{const} \neq 0.$$

For example:

$$\varepsilon + 100\varepsilon^2 = O(\varepsilon), \qquad \sin\varepsilon = O(\varepsilon), \qquad \frac{15\varepsilon}{2+4\varepsilon} = O(\varepsilon).$$

Similarly, one defines $O(\varepsilon^2)$, $O(\varepsilon^3)$, etc:

$$\lim_{\varepsilon\to 0}\frac{O(\varepsilon^n)}{\varepsilon^n} = \text{a nonzero number}.$$

The following "arithmetic" rules apply to the O-notation:

$$\begin{array}{rcl} O(\varepsilon)+O(\varepsilon) & = & O(\varepsilon) \\ O(\varepsilon)-O(\varepsilon) & = & O(\varepsilon) \\ \text{const}\cdot O(\varepsilon) & = & O(\varepsilon) \\ O(\varepsilon)\pm O(\varepsilon^2) & = & O(\varepsilon) \\ \varepsilon\cdot O(\varepsilon) & = & O(\varepsilon^2) \\ \frac{1}{\varepsilon}\cdot O(\varepsilon) & = & O(1). \end{array}$$

The generalization of these rules to other $O(\varepsilon^n)$ is obvious.

Thus, returning to our analysis, we can say that when $K \gg 1$,

$$y - y_0 = O\left(\frac{1}{K}\right).$$

Now let us consider the other limiting case.

$\underline{K \ll 1}$ Recall that we want to confirm that in the limit $K \to 0$, Eq. (3.10) reduces to Eq. (3.8b). To this end, use the Maclaurin series for e^x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

to expand the expression in the large parentheses in (3.10) up to $O(K^2)$ (you will see why in a moment):

$$\begin{array}{rcl} (1+Kv_0)\dfrac{1-(1-K\tau+\frac{K^2\tau^2}{2}+O(K^3))}{K}-\tau & = & \\ (1+Kv_0)\cdot\left(\tau-\dfrac{K\tau^2}{2}+O(K^2)\right)-\tau & = & \\ \tau-\dfrac{K\tau^2}{2}+O(K^2)+Kv_0\tau+O(K^2)+O(K^3)-\tau & = & \\ K\left(v_0\tau-\dfrac{\tau^2}{2}\right)+O(K^2). & & \end{array}$$

Substituting this back into (3.10), we find:

$$\begin{aligned} y &= y_0 + \frac{1}{gK}\left(K\left(v_0\tau - \frac{\tau^2}{2}\right) + O(K^2)\right) \\ &= y_0 + \frac{1}{g}\left(v_0\tau - \frac{\tau^2}{2}\right) + O(K). \end{aligned}$$

Thus, indeed, the $O(1)$-term in the above expression coincides with (3.8b). This indicates that our answer (3.10) is, most likely, correct.

3.1.3 Answering the main question

To conclude this section, let us answer the question posed in the title of this lecture, for model (3.5) (and hence for its solution (3.9), (3.10)).

It is easy to find the time, τ_m, needed for the ball to reach the maximum elevation. At its highest point, the ball has zero speed; then setting $v = 0$ in (3.9), one obtains (verify):

$$\tau_m = \frac{1}{K}\ln(1 + Kv_0). \tag{3.11}$$

However, it is not possible to find analytically the time τ_h when the ball *h*its the ground. This occurs when $y(\tau) = 0$, and it is not possible to solve analytically the transcendental equation (3.10) whose l.h.s. is set to 0.

Nonetheless, we can still answer our question by evaluating $y(2\tau_m)$. Indeed, if $y(2\tau_m) > 0$, then the ball is still in the air when $\tau = \tau_m + \tau_m$, i.e. going down is slower than going up. If, on the other hand, $y(2\tau_m) < 0$, then the ball has already hit the ground before $\tau = \tau_m + \tau_m$, so that in this case we would conclude that going down is faster. We present a somewhat technical calculation in Appendix B. It shows that $y(2\tau_m) > 0$, and hence it takes the ball longer to fall than to go up.

3.2 Perturbative treatment of model (3.5)

The perturbative treatment of the above model can be motivated by two different observations.

First, we note that expression (3.10) is rather cumbersome. In practice, the air resistance is quite small, and so all we really need is the first-order correction to the equations of motion (3.8) without the air resistance. To obtain such a correction, we need to expand (3.10) keeping higher orders of K than we did above when considering the limiting case $K \ll 1$.

Second, note that the air resistance could have been given by a more complicated function of v than (3.2), in which case it would not be possible to obtain an exact analytical solution for the counterpart of model (3.5). Yet, as long as the air resistance is small, we could hope to find *an approximate* solution of that model as being a *perturbation* of the solution (3.8) without the air resistance.

Below we illustrate the ***two approaches*** described in the above two paragraphs. In the ***first approach***, we seek a representation of solution (3.10) as:

$$y(\tau) = y^{(0)}(\tau) + Ky^{(1)}(\tau) + O(K^2),$$

where $y^{(0)}(\tau)$ is the resistance-free solution (3.8*b*) and $y^{(1)}$ is the first-order correction to it. To find $y^{(1)}$, we repeat the calculations done before Eq. (3.11), but keep one more power of K:

$$\begin{aligned}
y(\tau) &= \frac{1}{gK}\left((1+Kv_0)\frac{1-(1-K\tau+\frac{K^2\tau^2}{2}-\frac{K^3\tau^3}{6}+O(K^4))}{K} - \tau\right) \\
&= \frac{1}{gK}\left((1+Kv_0)\left(\tau - \frac{K\tau^2}{2} + \frac{K^2\tau^3}{6} + O(K^3)\right) - \tau\right) \\
&= \frac{1}{gK}\left(\tau - \frac{K\tau^2}{2} + \frac{K^2\tau^3}{6} + O(K^3) + Kv_0\tau - \frac{K^2 v_0\tau^2}{2} + O(K^3) - \tau\right) \\
&= \frac{1}{gK}\left(K\left(v_0\tau - \frac{\tau^2}{2}\right) + K^2\left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6}\right) + O(K^3)\right) \\
&= \frac{1}{g}\left(\left(v_0\tau - \frac{\tau^2}{2}\right) + K\left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6}\right) + O(K^2)\right).
\end{aligned}$$

Thus, we have obtained that

$$y(\tau) = \frac{1}{g}\left(\left(v_0\tau - \frac{\tau^2}{2} \right) + K\left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6} \right) + O(K^2) \right). \tag{3.12}$$

Question: When is this perturbative solution valid?
Answer: We will begin answering this with stating the evident fact that the perturbative solution must be valid when the correction term is much smaller than the drag-free term, i.e.

$$\left| K\left(-\frac{v_0\tau^2}{2} + \frac{\tau^3}{6} \right) \right| \ll \left| v_0\tau - \frac{\tau^2}{2} \right|. \tag{3.13}$$

For (3.13) to hold, it suffices that each term on the l.h.s. be smaller than each term on the r.h.s.. This occurs when:

$$Kv_0\tau^2/2 \ll v_0\tau; \quad K\tau^3/6 \ll v_0\tau; \qquad\qquad Kv_0\tau^2/2 \ll \tau^2/2; \quad K\tau^3/6 \ll \tau^2/2.$$

To continue, we need a new notation: the symbol "$\sim$", which means "equals in the order of magnitude sense". For example, one can write $1 \sim 2$ or $1 \sim 3$, i.e. this new notation allows us to ignore a factor such as 2 or 3 in our formulae.[4] Using this "order-of-magnitude" concept and cancelling by v_0 and τ in the preceding inequalities, we obtain three conditions (verify):

$$Kv_0 \ll 1, \qquad K\tau \ll 1, \qquad \text{and} \qquad K\tau^2 \ll v_0\,. \tag{3.14a}$$

These inequalities involve τ and v_0 and hence represent *two types* of conditions. One type, related to the initial velocity v_0, determines under what restrictions on the *initial conditions* the perturbative solution is valid. The other type, related to τ, determines *for what times* the perturbative solution will remain valid. In general, these two types of conditions may lead to *two different* restrictions under which (3.12) is valid. However, in the problem at hand, we will show below that both types of the condition lead to the *same* restriction. (In other words, we will show that all three of inequalities (3.14a) are equivalent.) Note that this is *not the general case*: In Lecture 8 you will encounter a situation when two types of conditions, involving the time and the parameters of the problem, will lead to two *different* restrictions.

We will continue answering the above Question by accepting the first condition in (3.14), i.e.

$$Kv_0 \ll 1, \tag{3.14b}$$

as valid, and will then show that it is equivalent to the other two. Note that this first condition restricts the *range of initial velocities* when one can apply our perturbative treatment. Indeed, if the Kv term in (3.5) is not much smaller than the first term on the r.h.s., then the drag's contribution is comparable to that of gravity, and hence there is no reason to treat the drag-related terms as small in (3.10).

Now let us turn to the second condition in (3.14a):

$$K\tau \ll 1. \tag{3.14c}$$

This is clearly the type of condition that restricts *the times* at which we consider our perturbative solution. However, these times also have the *natural restriction*, independent of whether

[4] A legitimate question to ask would be: Is $1 \sim 10$? An answer depends on particular circumstances. E.g., $1 \sim 10$ if we compare both these numbers with 1000, but $1 \not\sim 10$ if we compare them with 20.

the drag is small or not. Namely, the ball must remain in the air. Thus, we need to first look at what this natural restriction implies about τ and then account for the given information that the drag is small. If we consider just the upward motion of the ball, then we have $\tau \le \tau_m$, where τ_m is defined in (3.11). Apart from a very short part of the trajectory where $\tau \approx 0$, this implies

$$\tau \sim \tau_m;$$

i.e. we are ignoring the difference of these two quantities if it is a factor of about 2 or 3 or so. If we now use the assumption that the drag is small compared to the gravity, which is equivalent to condition (3.14b), one can estimate τ_m by its value *without* drag. That value is found from Eq. (3.8a) where one sets $v = 0$ on the l.h.s. (similarly to the derivation of (3.11)). The result is: $\tau_{m,\,\text{nodrag}} = v_0$. Along with the last displayed formula above this implies:

$$\tau \sim v_0 \,.$$

Using this order-of-magnitude estimate, it is now clear that not only the first two inequalities in (3.14a) (i.e., (3.14b) and (3.14c)), but also the third one, are all equivalent for the *upward* motion.

For the *downward* motion, it suffices to note that $\tau_m < \tau \le \tau_h$, where τ_h is the time when the ball hits the ground. In the case of a small drag, $\tau_h - \tau_m \approx \tau_m$ (see the second sentence of this Lecture). Then for the downward motion one has $\tau_m < \tau \lesssim 2\tau_m$, i.e., again, $\tau \sim \tau_m$. Then the same considerations as in the previous paragraph lead to the conclusion that the three conditions in (3.14a) are all equivalent and, in particular, are equivalent to (3.14b).

Now, let us explore the ***second approach*** described at the beginning of this section. Recall that the first approach has relied on knowing the exact solution (3.10) and Taylor-expanding it into the approximate form (3.12). In contrast, the second approach will not require the knowledge of the exact solution and hence can be employed for *any* law of air resistance, as long as the drag is small compared to gravity. Thus, consider model (3.5) where the term $Kv \ll 1$. Note that this is precisely the condition under which the perturbative solution is valid; however, in this case, it arises from purely physical consideration that the air resistance be small compared to the gravity.

Let us seek the solution $v(\tau)$ in the form:

$$v = v^{(0)} + Kv^{(1)} + K^2 v^{(2)} + \dots, \tag{3.15}$$

where $v^{(0)}$, $v^{(1)}$, $v^{(2)}$, etc. do not depend on K. Substituting (3.15) into (3.5) we obtain:

$$\dot{v}^{(0)} + K\dot{v}^{(1)} + O(K^2) = -1 - K\big(v^{(0)} + Kv^{(1)} + O(K^2)\big).$$

Let us now collect the terms at like powers of K:

at K^0:

$$\dot{v}^{(0)} = -1.$$

This is the equation for the resistance-free case, as expected. With the initial condition from (3.7), we have:

$$v^{(0)} = v_0 - \tau$$

(which, of course, is (3.8a)). Next,

at K^1:

$$
\begin{aligned}
\dot{v}^{(1)} &= -v^{(0)} \quad \overset{\text{(see the equation above)}}{\Rightarrow} \\
\dot{v}^{(1)} &= -v_0 + \tau \quad \Rightarrow \\
v^{(1)} &= v^{(1)}(\tau = 0) - v_0\tau + \frac{\tau^2}{2} \\
&= -v_0\tau + \frac{\tau^2}{2}.
\end{aligned}
$$

Here we have again used the initial condition (3.7), which implies that since $v(\tau = 0) = v_0$ and since we have taken $v^{(0)}(0) = v_0$, then $v^{(n)}(0) = 0$ for $n = 1, 2, \ldots$. Substituting the expressions for $v^{(0)}$ and $v^{(1)}$ into (3.15), we find:

$$
v = (v_0 - \tau) - K\left(v_0\tau - \frac{\tau^2}{2}\right) + O(K^2), \tag{3.16a}
$$

and hence (using $y_0 = 0$ and omitting the $O(K^2)$-term):

$$
y = \frac{1}{g}\left(\left(v_0\tau - \frac{\tau^2}{2}\right) - K\left(\frac{v_0\tau^2}{2} - \frac{\tau^3}{6}\right)\right). \tag{3.16b}
$$

This is the same as (3.12), as it should be.

Thus, we have shown that the same perturbative solution can be obtained by two independent approaches: by Taylor-expanding the exact solution and by perturbatively solving the model, Eq. (3.5). In most practical cases, when the exact solution is not available, the second approach may be the only one that can give an approximate solution.

To conclude this section, let us find the perturbative expressions for the times of going up and down, and thereby confirm our earlier conclusion that going down takes longer. First, from (3.16a), we find the time of going up as the particular value of τ when $v = 0$:

$$
0 = v_0 - \tau_m - Kv_0\tau_m + \frac{K\tau_m^2}{2}, \tag{3.17}
$$

where we have omitted the $O(K^2)$-term. This is a quadratic equation for τ and can be solved exactly. However, a *much easier* approach is to seek the solution τ_m in the form similar to (3.15):

$$
\tau_m = \tau_m^{(0)} + K\tau_m^{(1)} + O(K^2). \tag{3.18}
$$

Substituting (3.18) into (3.17), we obtain:

$$
0 = v_0 - (\tau_m^{(0)} + K\tau_m^{(1)} + O(K^2)) - Kv_0(\tau_m^{(0)} + K\tau_m^{(1)} + O(K^2)) + \frac{K}{2}(\tau_m^{(0)} + K\tau_m^{(1)} + O(K^2))^2.
$$

Collecting terms at like powers of K:

at K^0:

$$
0 = v_0 - \tau_m^{(0)} \quad \Rightarrow \quad \tau_m^{(0)} = v_0.
$$

at K^1:

$$
0 = -\tau_m^{(1)} - v_0\tau_m^{(0)} + \frac{(\tau_m^{(0)})^2}{2} \quad \Rightarrow \quad \tau_m^{(1)} = -v_0\tau_m^{(0)} + \frac{(\tau_m^{(0)})^2}{2} = -\frac{v_0^2}{2}.
$$

Thus,

$$\tau_m = v_0 - \frac{Kv_0^2}{2} + O(K^2). \tag{3.19}$$

Verify that this agrees with the first two terms of the expansion of (3.11) when $K \ll 1$ (use the Maclaurin series stated after Eq. (3.14)).

Now let us use the same method to find the time, τ_h, when the ball hits the ground. Substituting into (3.16*b*) with $y = 0$ an expansion

$$\tau_h = \tau_h^{(0)} + K\tau_h^{(1)} + O(K^2),$$

we find, omitting $O(K^2)$ terms:

$$0 = v_0(\tau_h^{(0)} + K\tau_h^{(1)}) - \frac{1}{2}(\tau_h^{(0)} + K\tau_h^{(1)})^2 - K\left[\frac{v_0}{2}(\tau_h^{(0)} + K\tau_h^{(1)})^2 - \frac{1}{6}(\tau_h^{(0)} + K\tau_h^{(1)})^3\right]$$

Collecting the coefficients at like powers of K:

at K^0:

$$0 = v_0\tau_h^{(0)} - \frac{1}{2}(\tau_h^{(0)})^2 \qquad \Rightarrow \qquad \tau_h^{(0)} = 2v_0.$$

at K^1:

$$0 = v_0\tau_h^{(1)} - \frac{1}{2}\cdot 2\tau_h^{(0)}\tau_h^{(1)} - \left[\frac{v_0}{2}\cdot(\tau_h^{(0)})^2 - \frac{1}{6}(\tau_h^{(0)})^3\right].$$

IMPORTANT NOTE: Although the original equation for τ_h was nonlinear (see (3.16*b*) with $y = 0$), the equation for the correction $\tau_h^{(1)}$ (and for all higher-order corrections $\tau_h^{(2)}$, $\tau_h^{(3)}$, etc., if we decide to find them) is *linear*, and hence can always be solved and yields a unique solution.

Continuing, from the above equation we have:

$$\tau_h^{(1)}(v_0 - \tau_h^{(0)}) = \frac{v_0}{2}(\tau_h^{(0)})^2 - \frac{1}{6}(\tau_h^{(0)})^3,$$

and, using the above expression for $\tau_h^{(0)}$:

$$\tau_h^{(1)} = -\frac{2}{3}v_0^2$$

(verify). Thus,

$$\tau_h = 2v_0 - K\cdot\frac{2}{3}v_0^2 + O(K^2). \tag{3.20}$$

From (3.19) and (3.20), the time required for the ball to go down is:

$$\tau_h - \tau_m = 2v_0 - \frac{2K}{3}v_0^2 + O(K^2) - v_0 + \frac{K}{2}v_0^2 - O(K^2) \;=\; v_0 - K\cdot\frac{1}{6}v_0^2 + O(K^2). \tag{3.21}$$

Comparing (3.21) with (3.19), we see that the time to go down is greater than the time to go up, as was proved in general in Section 3.1.

3.3 Model with quadratic air resistance

We will now follow the steps of Sections 3.1 and 3.2 to analyze the solution of the model with the air resistance force given by (3.3):

$\underline{v > 0}$

$$\dot{v} = -1 - Kv^2\,, \tag{3.22a}$$

$\underline{v < 0}$

$$\dot{v} = -1 + Kv^2 \tag{3.22b}$$

(see (3.1) and (3.3)). (Note that physically, this model is not applicable to the motion of the ball, but the mathematical perturbation approach carries over to it without changes, and it is this approach that we intend to practice in this lecture.) I will go briefly over the main steps of the solution. You will be asked to supply the missing details in the homework. We have to analyze (3.22a) and (3.22b) separately, since these are different equations. Let us begin with (3.22a). The solution to (3.22a) and (3.7) is given by:

$$\frac{\arctan(\sqrt{K}v)}{\sqrt{K}} = -\tau + \frac{\arctan(\sqrt{K}v_0)}{\sqrt{K}}, \tag{3.23a}$$

$$y = y_0 + \frac{1}{gK}\ln\frac{\cos\left[\arctan(\sqrt{K}v_0) - \sqrt{K}\tau\right]}{\cos(\arctan(\sqrt{K}v_0))}\,. \tag{3.23b}$$

(Strictly speaking, one requires the absolute value sign of the argument of the logarithm, but one can show — which we will not do here — that this argument is always positive in our problem, and hence no $|\dots|$ is needed.) The exact time to reach the highest point is found from (3.23a):

$$\tau_m = \frac{\arctan(\sqrt{K}v_0)}{\sqrt{K}}. \tag{3.24}$$

As in Section 3.2, here our goal will be to obtain the perturbation-type solution for v, y, and τ_m by using both approaches presented there. We will begin with the "first approach", where we will use Taylor expansion of the exact solutions (3.23) and (3.24), and then will re-obtain the same results using the "second approach", where we will use an expansion analogous to (3.15) to solve (3.22a) perturbatively.

Let us start with the "first approach" of Section 3.2. We will, however, employ a step from the "second approach" to simplify its calculations; let us explain this. It is, of course, possible to solve (3.23a) for v and Taylor-expand the answer to obtain an expression for $v(\tau)$ of the form (3.15). Similarly, it is possible to Taylor-expand (3.23b) to get a perturbative expression for $y(\tau)$. However, this would be significantly more cumbersome to do than to obtain the same results by the method described in the next paragraph. In homework Problem 1 you will be asked to follow this latter (and easier) method.

The "simplified first approach" proceeds as follows. To obtain the approximate solution of (3.23a), first note that the arguments of both arctangents are small because $K \ll 1$. Then, use the Maclaurin expansion of $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + O(x^5), \quad x \ll 1$$

to expand both arctangents. Finally, substitute expansion (3.15) into the l.h.s. of the so-expanded Eq. (3.23a) and follow the lines of the derivation of Eq. (3.19) from (3.17) to find:

$$v = (v_0 - \tau) + K\left(-v_0^2\tau + v_0\tau^2 - \frac{\tau^3}{3}\right) + O(K^2). \tag{3.25a}$$

Now, instead of Taylor-expanding the r.h.s. of (3.23b) (which would have been very cumbersome!), integrate (3.25a) to obtain (with $y_0 = 0$):

$$y = \frac{1}{g}\left[(v_0\tau - \frac{\tau^2}{2}) + K\left(-\frac{v_0^2\tau^2}{2} + \frac{v_0\tau^3}{3} - \frac{\tau^4}{12}\right) + O(K^2)\right]. \tag{3.25b}$$

When are the perturbative expansions (3.25a) and (3.25b) valid? See a similar discussion after Eq. (3.12).

Finally, from (3.24), obtain:

$$\tau_m = v_0 - K\frac{v_0^3}{3} + O(K^2). \tag{3.26}$$

Thus, you have found the perturbative solution (3.25) and (3.26) by the "simplified first approach" of Section 3.2.

Let us now re-obtain the above results by the "second approach" of Section 3.2, which starts with Eq. (3.15). First, substitute expansion (3.15) into (3.22a) and follow the steps of derivation of (3.16a) to re-obtain (3.25a). (You do not need to re-obtain (3.25b).) Then, to conclude the treatment of the upward motion of the ball, re-obtain (3.26) starting with (3.25a). Use expansion (3.18) for τ_m and follow the approach presented after that equation.

Now, turn to the downward motion of the ball, described by Eq. (3.22b). As before, begin by finding its exact solution. In an implicit form, it is:

$$\frac{1}{2\sqrt{K}}\ln\left|\frac{1+\sqrt{K}v}{1-\sqrt{K}v}\right| = -\tau + C, \tag{3.27a}$$

where C is the integration constant. At $\tau = \tau_m$ (i.e. when the ball is at the highest point of its trajectory), $v = 0$. From this condition, deduce the value of C, by setting $v = 0$ and $\tau = \tau_m$ in (3.27a).

Next, solve (3.27a) for v to obtain:

$$v = -\frac{1}{\sqrt{K}}\cdot\frac{1-\exp\left(-2\sqrt{K}(\tau - C)\right)}{1+\exp\left(-2\sqrt{K}(\tau - C)\right)}. \tag{3.27b}$$

(Physically, what does the minus sign in front of this expression tell you?) To conclude solving for v, transform (3.27b) into:

$$v = -\frac{1}{\sqrt{K}}\tanh\left(\sqrt{K}(\tau - C)\right). \tag{3.28a}$$

To obtain $y(\tau)$, integrate (3.28a) and use the intial condition that at $\tau = \tau_m$, the ball is at its highest elevation. Let us denote this elevation y_m. Then, obtain from (3.28a) that

$$y = y_m - \frac{1}{gK}\ln\left[\cosh\left(\sqrt{K}(\tau - C)\right)\right]. \tag{3.28b}$$

Finally, find $y_m \equiv y(\tau_m)$ from (3.23*b*) (assuming that $y_0 = 0$). When you put all these results together, your answer for $y(\tau)$ should be equivalent to

$$y = \frac{1}{gK} \ln \frac{\sqrt{1+Kv_0^2}}{\cosh\left(\sqrt{K}\tau - \arctan(\sqrt{K}v_0)\right)}. \tag{3.29}$$

Interestingly enough, unlike for the model with the linear (in v) air resistance, for the model with the quadratic in v air resistance, it *is* possible to find the analytic expression for the time when the ball hits the ground. By setting $y = 0$ in (3.29), show that this time, τ_h, is given by:

$$\tau_h = \tau_m + \frac{1}{\sqrt{K}} \ln\left(\sqrt{1+Kv_0^2} + \sqrt{K}v_0\right). \tag{3.30}$$

Here you need to make use of the identity

$$\operatorname{arccosh} x = \ln\left(x + \sqrt{x^2-1}\right).$$

This concludes finding the exact solution for the going-down case.

From this point on, repeat the steps you did for the going-up case. That is, you will first obtain approximate solutions from the exact ones by Taylor-expanding the latter. To start, obtain the Taylor expansions of the l.h.s. of (3.27*a*), valid up to terms $O(K)$ (i.e., obtain the expression for the coefficient of the $O(K)$-term). To do so: (i) split the logarithm into two terms using a property of logarithms, (ii) expand the logarithms into Maclaurin series (use the formula in a homework problem), and then (iii) substitute expansion (3.15) and use the approach of obtaining Eq. (3.19). The result should be:[5]

$$v = -\left((\tau - C) - \frac{K}{3}(\tau - C)^3 + O(K^2)\right). \tag{3.31a}$$

Next, integrate the last expression and obtain an $O(K)$-accurate expression for y:

$$y = y_m - \frac{1}{g}\left(\frac{(\tau-C)^2}{2} - \frac{K(\tau-C)^4}{12} + O(K^2)\right). \tag{3.31b}$$

In (3.31*a*) and (3.31*b*), substitute the expression for C which you have found earlier. (You do *not* need to Taylor-expand that expression for C in powers of K; leave it as is.) Then derive an $O(K)$-accurate expression for τ_h from (3.30):

$$\tau_h = \tau_m + \left(v_0 - \frac{Kv_0^3}{6} + O(K\sqrt{K})\right); \tag{3.31c}$$

a technical comment about the $O(K\sqrt{K})$-term in (3.31*c*) is found in the footnote[6]. This concludes the step of obtaining approximate solutions from the exact solutions (3.27)–(3.30).

When are the expansions (3.31) valid?

[5] It is actually easier to obtain (3.31*a*) by Taylor-expanding the explicit solution (3.28*a*) than the implicit solution (3.27*a*). The reason that you are asked to use the more difficult approach here is that it is more advanced and also more general. In particular, it does not rely on the possibility to solve an implicit equation, like (3.27*a*), for an explicit answer.

[6] It is possible to show that it is actually $O(K^2)$, but this is more difficult to do; so just show that this term is no greater than $O(K\sqrt{K})$.

Finally, re-obtain the approximate solutions (3.31) starting from the differential equation (3.22*b*). To begin, re-obtain (3.31*a*) by substituting expansion (3.15) into (3.22*b*). Once you have obtained (3.31*a*), it can be integrated to yield (3.31*b*). You do not need to do this integration here because you have already done it above when obtaining (3.31*b*) for the first time. Now, to conclude, use the expansion for $(\tau_h - \tau_m)$ similar to the one found after Eq. (3.19), i.e.:

$$\tau_h - \tau_m = \Delta\tau^{(0)} + K\Delta\tau^{(1)} + O(K^2), \tag{3.32}$$

to re-obtain (3.31*c*) from (3.31*b*). Further details of this step are given in a homework problem.

3.4 Appendix A: Two ways to nondimensionalize Eq. (3.4)

3.4.1 Way 1

We will use dimensional parameters of Eq. (3.4) to construct a parameter with the dimension of time, which we will denote t_{scale}. Let us notice that since l.h.s. of (3.4) has the dimension [velocity]/[time], then so do terms on the r.h.s.. Then from the second term there we see that the coefficient D/m must have the dimension 1/[time], and hence the desired combination of parameters is:

$$t_{\text{scale}} = m/D. \tag{3.33}$$

Then, in similarity with how we derived (3.5), we introduce a new variable

$$\tau = t/t_{\text{scale}}. \tag{3.34}$$

Note that unlike in Section 3.1, where τ had the dimension of velocity (verify), here our new τ is *nondimensional*. Proceeding as in the derivation of (3.5), we arrive at:

$$\dot{v} = -g t_{\text{scale}} - \frac{D}{m} t_{\text{scale}}\, v \qquad \Rightarrow \qquad \dot{v} = -(mg/D) - v, \tag{3.35}$$

where in the last step we have used (3.33).

Equation (3.35) has two coefficients on the r.h.s.: mg/D and 1 (the latter multiplies v), each of which has a specific meaning. The former coefficient is the so-called terminal velocity:

$$v_{\text{term}} = mg/D. \tag{3.36}$$

This is the velocity that is asymptotically achieved by the ball that is allowed to fall infinitely (or just very) long; it results from the balance between the gravity and the air drag. When this balance is achieved, the velocity stops changing: one has $dv/dt = 0$ in (3.4), from which (3.36) is then obtained.

The second coefficient, 1, on the r.h.s. of (3.35), says that in the nondimensional time (3.34), the drag causes the velocity to change with the rate of order one. This is *not* a convenient normalization when the air resistance is small compared to gravity, since it would make sense to normalize the equations so that it will be the gravity that will cause order one changes in the velocity, while the drag will cause a much smaller rate of change of v. In the next subsection we present an alternative way to nondimensionalize Eq. (3.4) so as to satisfy this condition.

3.4.2 Way 2

As we have pointed out in the previous paragraph, our new time scale must not involve the drag coefficient D. One cannot construct a quantity with the dimension of time from the remaining two parameters g and m. A hint of what one should do comes from the following consideration. Suppose we throw the ball up into the air with the velocity v_0. It will stop in time

$$t_{\rm up} = v_0/g, \tag{3.37a}$$

where we have neglected the air resistance. The time defined by this equation is the time scale where gravity plays the main role. This is precisely our goal, and therefore we will define the new time scale to be as in (3.37a), i.e.:

$$t_{\rm scale} = v_0/g, \tag{3.37b}$$

Defining a new $\tau = t/t_{\rm scale}$ and proceeding as in the previous subsection, we obtain

$$\dot{v} = v_0 - (v_0/v_{\rm term})\, v. \tag{3.38}$$

From this equation it becomes transparent that the effect of the air drag is proportional to the ratio of the initial and terminal velocities.

Finally, one can completely nondimensionalize (3.38) by dividing it through by v_0. Then, for the new nondimensional velocity $u = v/v_0$ (i.e velocity measured relative to its initial value) one obtains:

$$\dot{u} = 1 - (v_0/v_{\rm term})\, u. \tag{3.39}$$

3.5 Appendix B: Showing that $y(2\tau_m) > 0$ in Section 3.1.3

Setting $y_0 = 0$ in (3.10) and using τ_m from (3.11), we find:

$$y(2\tau_m) = \frac{1}{gK}\left((1+Kv_0)\frac{1-(e^{-K\tau_m})^2}{K} - 2\tau_m \right)$$

$$\overset{\text{use (3.11)}}{=} \frac{1}{gK}\left(\frac{1+Kv_0}{K}\cdot\left(1 - \frac{1}{(1+Kv_0)^2}\right) - \frac{2}{K}\ln(1+Kv_0) \right).$$

Let us denote $1 + Kv_0 \equiv x(>1)$. Then $y(2\tau_m) = \left(x - \frac{1}{x} - 2\ln x\right)/(gK^2)$ (verify). The r.h.s. of this expression is a function of x, $f(x)$:

$$f(x) = x - \frac{1}{x} - 2\ln x.$$

It is easy to see that:

$$f(1) = 1 - 1 - 2\ln 1 = 0, \quad \text{and}$$

$$f'(x) = 1 + \frac{1}{x^2} - \frac{2}{x} = \left(1 - \frac{1}{x}\right)^2 > 0, \qquad \text{for } x > 1.$$

Therefore, $f(x)$ increases, and so $f(x) > 0$ for $x > 1$. Thus,

$$y(2\tau_m) > 0,$$

and hence it takes the ball longer to fall down than to go up.