4 Will a car fly off the rollercoaster?

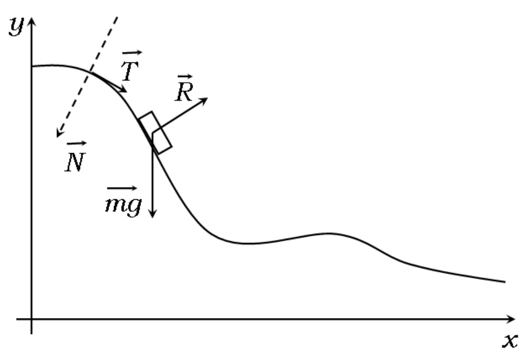
In this lecture, we will use the motion of an object (a car) sliding down a rollercoaster to illustrate three concepts. First, we will show a simple application of the equations of motion along a curve to a real-world problem. Second, we will show how the Chain Rule is used when a variable in a differential equation is replaced with a related variable. Third and last, we will see that when a differential equation is too complicated to be solved analytically, it can always be solved numerically.

Consider the motion of a car sliding down a rollercoaster whose shape is given by $y = f(x)$. Our goal is to derive the equations of the motion and, eventually, be able to answer the following question: If there is a bump somewhere on the track, will the car go flying off into the air or will it stay on the rollercoaster? In this Lecture, we will neglect friction of any sort; in a homework problem you will be asked to show how it can be included.

The motion of the car is governed by Newton's Second Law:

$$m\vec{a} = m\vec{g} + \vec{R}, \tag{4.1}$$

where $\vec{R}$ is the normal reaction force. The objective of this Lecture is to find $\vec{a}$ and $\vec{R}$.



Let us choose the coordinate system that at any given moment coincides with the tangent and normal vectors, $\vec{T}$ and $\vec{N}$, at the location of the car. (In the figure above, $\vec{T}$ and $\vec{N}$ are shown at a different location than the car so as not to congest the figure.)

As its name suggests, the *normal* reaction force $\vec{R}$ is normal to the curve and hence lies on the line containing $\vec{N}$; thus, $\vec{R} \parallel \vec{N}$.

In Calculus III you learned that the acceleration can be expanded along $\vec{T}$ and $\vec{N}$ as follows:

$$\vec{a} = a_T\vec{T} + a_N\vec{N}, \tag{4.2}$$

$$a_T = \frac{d^2 s}{dt^2}, \qquad a_N = K(s)\left(\frac{ds}{dt}\right)^2 .$$

Here s is the arclength from, say, the starting point of the motion to the current location of the car, and $K(s)$ is the curvature at that location. Recall the meaning of (4.2): it says that the total acceleration is a vector sum of the linear acceleration, d^2s/dt^2, pointing along the direction of motion, and the centripetal acceleration, $K(s)(ds/dt)^2$, pointing towards the center of the curvature. Indeed,

$$K(s)\left(\frac{ds}{dt}\right)^2 = \frac{(ds/dt)^2}{1/K(s)} = \frac{(\text{velocity})^2}{\text{radius of curvature}},$$

which is the formula for the centripetal acceleration of a uniform circular motion, as studied in an elementary Physics course. We will require a formula for $K(s)$ derived in Calculus III:

$$K = \left\| \frac{d^2\vec{r}}{ds^2} \right\| = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2}. \tag{4.3}$$

Let us now project Eq. (4.1) onto the axes $\vec{T}$ and $\vec{N}$. This is done by taking the dot product of (4.1) with $\vec{T}$ and $\vec{N}$. The operation of taking dot product will be denoted by $\bullet$ in this Lecture. We will use identity $\vec{T} \bullet \vec{T} = \|\vec{T}\|^2 = 1$ for the unit vector $\vec{T}$, and similarly for $\vec{N}$.

Projecting on $\vec{T}$, we have:

$$(a_T \vec{T} + a_N \vec{N}) \bullet \vec{T} = \vec{g} \bullet \vec{T} + \frac{1}{m} \vec{R} \bullet \vec{T}.$$

Using the orthogonality of $\vec{T}$ to $\vec{N}$ and hence to $\vec{R}$ (which implies $\vec{T} \bullet \vec{N} = \vec{T} \bullet \vec{R} = 0$), we find:

$$a_T = \vec{g} \bullet \vec{T}. \tag{4.4a}$$

Projecting on $\vec{N}$, we similarly obtain:

$$a_N = \vec{g} \bullet \vec{N} + \frac{1}{m} \vec{R} \bullet \vec{N}. \tag{4.4b}$$

We will now use the two Eqs. (4.4) to derive equations of motion for the car. Actually, Eq. (4.4a) already has the form of such an equation:

$$\frac{d^2 s}{dt^2} = \text{some function of } s,$$

where we have used the expression for a_T found after (4.2) and the fact that $\vec{T}$ is uniquely characterized by the location of the car along the track, i.e., by s. However, using s as the variable characterizing the car's location is *not* convenient. Indeed, if we know s, we still have to find the x- and y-coordinates of the car, and this is not an easy task. From Calculus II, one knows that the formula for the arclength is:

$$s(x) = \int_{x_0}^{x} \sqrt{1 + \left(\frac{dy(\tilde{x})}{d\tilde{x}} \right)^2} \, d\tilde{x} \tag{4.5}$$

(here $\tilde{x}$ is the dummy integration variable). Therefore, to solve for x given a value of s, one would need to: (i) do the integration in (4.5), and then (ii) solve the (typically nonlinear) algebraic equation $s(x) = s$ for x. Both of these steps can rarely be accomplished analytically.

On the other hand, using x to characterize the car's location *is* convenient. Indeed, if we know the car's x-coordinate, we can also immediately find its y-coordinate as $y = y(x)$, where the latter function describes the shape of the rollercoaster. Thus, we will derive equations of the car's motion in terms of its x-coordinate, where $x = x(t)$.

To that end, we need to switch from the variable s in $(4.4a, b)$ to x. We will start with $(4.4a)$ and will consider its left- and right-hand sides separately.

L.h.s. of $(4.4a)$:

$$\frac{d^2 s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) \overset{\text{Chain Rule}}{=} \frac{d}{dt}\left(\frac{ds}{dx} \cdot \frac{dx}{dt}\right).$$

Next, by the Fundamental Theorem of Calculus,

$$\frac{ds}{dx} \overset{(4.5)}{=} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \equiv Q(x). \tag{4.6}$$

Continuing with the equation for d^2s/dt^2, we have:

$$\begin{aligned}\frac{d^2s}{dt^2} &= \frac{d}{dt}\left(Q\cdot\frac{dx}{dt}\right) = \frac{dQ}{dt}\cdot\frac{dx}{dt}+Q\frac{d^2x}{dt^2}\\ &\overset{\text{Chain Rule}}{=} \left(\frac{dQ}{dx}\cdot\frac{dx}{dt}\right)\frac{dx}{dt}+Q\frac{d^2x}{dt^2} = Q'\left(\frac{dx}{dt}\right)^2+Q\frac{d^2x}{dt^2}.\end{aligned}\tag{4.7}$$

Here and below, we use the notations

$$Q'\equiv\frac{dQ}{dx},\qquad y'\equiv\frac{dy}{dx}.$$

R.h.s. of (4.4a):

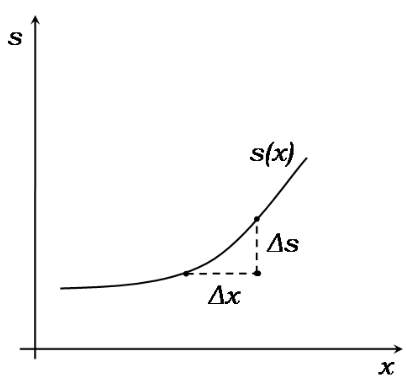
$$\vec{g}=\langle 0,-g\rangle$$

$$\vec{T}\overset{\text{Calculus III}}{=}\frac{d\vec{r}}{ds}\equiv\frac{d\langle x,y\rangle}{ds}=\left\langle\frac{dx}{ds},\frac{dy}{ds}\right\rangle,\qquad\Rightarrow$$

$$\vec{g}\bullet\vec{T}=-g\frac{dy}{ds}\overset{\text{Chain Rule}}{=}-g\frac{dy}{dx}\cdot\frac{dx}{ds}=-g\cdot\frac{y'}{ds/dx}\overset{(4.6)}{=}-g\cdot\frac{y'}{Q}.$$

Here we have used the fact that if $s(x)$ and $x(s)$ are the inverse functions of each other, then

$$\frac{ds}{dx}=\frac{1}{dx/ds}.\tag{4.8}$$



The reason for this is illustrated in the figure on the left, where one can see that:

$$\frac{ds}{dx}\approx\frac{\Delta s}{\Delta x}=\frac{1}{\Delta x/\Delta s}\approx\frac{1}{dx/ds}.$$

Note that *no* analogues of (4.8) hold for higher derivatives, i.e.

$$\frac{d^2s}{dx^2}\neq\frac{1}{d^2x/ds^2},\quad\text{etc.,}\tag{4.9}$$

as we will explicitly demonstrate later.

We now put all the pieces above together to transform (4.4a) to the form:

$$Q\frac{d^2x}{dt^2}+Q'\left(\frac{dx}{dt}\right)^2=-g\cdot\frac{y'}{Q}$$

$$\overset{\textbf{verify}}{\Rightarrow}$$

$$\frac{d^2x}{dt^2}=-\left(g\cdot\frac{y'}{Q^2}+\frac{Q'}{Q}\cdot\left(\frac{dx}{dt}\right)^2\right).\tag{4.10a}$$

We compute Q' using its definition (4.6):

$$Q'=\frac{d}{dx}\sqrt{1+(y')^2}=\frac{y'y''}{Q},\tag{4.11}$$

and substitute this into (4.10a) to obtain:

$$\frac{d^2x}{dt^2} = -\frac{y'}{Q^2}\left(g + y''\left(\frac{dx}{dt}\right)^2\right). \tag{4.10b}$$

We now turn to transforming (4.4b) similarly to how we transformed (4.4a) into (4.10a) or (4.10b).

L.h.s. of (4.4b)

As we have found above,

$$\left(\frac{ds}{dt}\right) = \frac{ds}{dx}\cdot\frac{dx}{dt} = Q\frac{dx}{dt}.$$

Now we need to transform the expression for $K(s)$, Eq. (4.3), into a function of x. First,

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\left(\frac{dx}{ds}\right) \overset{(4.8),\,(4.6)}{=} \frac{d}{ds}\left(\frac{1}{Q(x)}\right) = -\frac{1}{Q^2}\frac{dQ}{ds} = -\frac{1}{Q^2}\frac{dQ}{dx}\cdot\frac{dx}{ds} = -\frac{Q'}{Q^3}.$$

(If we now recall that $d^2s/dx^2 = Q'$ (see (4.6)), we note that we have explicitly verified (4.9).) Next,

$$\frac{d^2y}{ds^2} = \frac{d}{ds}\left(\frac{dy}{ds}\right) = \frac{d}{ds}\left(\frac{dy}{dx}\cdot\frac{dx}{ds}\right) = \frac{d}{ds}\left(\frac{y'}{Q}\right) = \frac{d}{dx}\left(\frac{y'}{Q}\right)\cdot\frac{dx}{ds} = \frac{y''Q - Q'y'}{Q^3}.$$

To simplify the above expressions, we substitute the formula for Q', which is given by Eq. (4.11), into the above expressions for d^2x/ds^2, and find (verify):

$$\frac{d^2x}{ds^2} = -\frac{y'y''}{Q^4}, \qquad \frac{d^2y}{ds^2} = \frac{y''}{Q^4}, \tag{4.12}$$

where in deriving the second formula we have used the definition of Q. Substituting (4.12) into (4.3), we find:

$$K = \sqrt{\left(\frac{y''}{Q^4}\right)^2 + \left(-\frac{y'y''}{Q^4}\right)^2} = \frac{|y''|}{Q^4}\sqrt{1+(y')^2} = \frac{|y''|}{Q^3}, \tag{4.13}$$

where we have used the fact: $\sqrt{a^2} = |a|$ for any real a. Thus, the l.h.s. of (4.4b) becomes:

$$K\cdot\left(\frac{ds}{dt}\right)^2 = \frac{|y''|}{Q}\left(\frac{dx}{dt}\right)^2. \tag{4.14}$$

R.h.s. of (4.4b)

To compute $\vec{g}\bullet\vec{N}$, one needs an expression for $\vec{N}$. In Calculus III, it is shown that

$$\vec{N} = \frac{d^2\vec{r}/ds^2}{K}.$$

Then

$$\begin{aligned}\vec{g}\bullet\vec{N} \quad &= \quad \langle 0, -g\rangle \bullet \left\langle \frac{d^2x}{ds^2}, \frac{dy^2}{ds^2}\right\rangle / K \\ &\overset{(4.12),\,(4.13)}{=} -g\cdot\frac{y''/Q^4}{|y''|/Q^3} = -g\cdot\frac{\mathrm{sgn}(y'')}{Q},\end{aligned} \tag{4.15}$$

where for any nonzero number α,

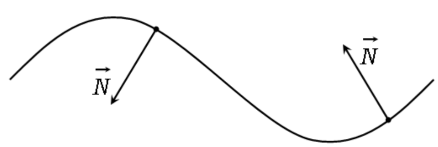
$$\text{sgn}(\alpha) = \begin{cases} +1, & \alpha > 0 \\ -1, & \alpha < 0. \end{cases}$$

In addition, we *define*

$$\text{sgn}(0) = +1.$$

(We could have also made the opposite choice, $\text{sgn}(0) = -1$; it would not affect the final result as long as we consistently stay with one definition.) The reason we need to define sgn(0) is to be able to handle the case of a flat inclined plane, where $y'' = 0$.

The other term on the r.h.s. of (4.4*b*) is $\vec{R} \bullet \vec{N}$. Since $\vec{R}$ and $\vec{N}$ lie on the same line (why?) and $\vec{N}$ is a unit vector, then $\vec{R} \bullet \vec{N}$ equals $+|R|$ or $-|R|$. To decide which sign we need to take, observe the following. As long as the car stays on *the top side* of the track and the track *has no loops*, the vertical component of $\vec{R}$ is positive. As a shorthand, we will say that "$\vec{R}$ is pointing upwards".



Next, as shown in Calculus III, $\vec{N}$ is always pointing towards the instantaneous center of curvature of the track (see the figure on the left). Thus, $\vec{N}$ is pointing upward when $y'' > 0$ and $\vec{N}$ is pointing downward when $y'' < 0$. When $y'' = 0$, we need to *define* $\vec{N}$ as pointing upward in order to be consistent with our own definition of sgn(0).

Combining these observations of $\vec{R}$ and $\vec{N}$, we have:

$$\vec{R} \bullet \vec{N} = \begin{cases} |R|, & y'' \geq 0 \\ -|R|, & y'' < 0 \end{cases} = |R| \, \text{sgn}(y''). \tag{4.16}$$

Let us stress that (4.16) holds under the two aforementioned assumptions: (i) the car is on the top side of the track, and (ii) the track has no loops. In a homework problem, you will consider how this analysis needs to be modified when the reverse of assumption (i) holds, namely, when the car is on the bottom side of the track. As far as lifting assumption (ii), this requires a somewhat more careful treatment, which we will not pursue.

Substituting (4.14), (4.15), (4.16) into (4.4*b*), we rewrite the latter equation as (verify):

$$|R| = m \left(\frac{g}{Q} + \frac{y''}{Q} \left(\frac{dx}{dt} \right)^2 \right). \tag{4.17}$$

We have used the identity $(\text{sgn}(\alpha))^2 = 1$.

To conclude our analysis, we will present Eqs. (4.10*b*) and (4.17) together, comment on their meaning, and then answer the question posed in the title of this Lecture. Thus:

$$\frac{d^2x}{dt^2} = -\frac{y'}{Q^2} \left(g + y'' \left(\frac{dx}{dt} \right)^2 \right), \tag{4.10b}$$

$$|R| = \frac{m}{Q}\left(g + y''\left(\frac{dx}{dt}\right)^2\right). \tag{4.17}$$

Equation (4.10*b*) is a differential equation for $x(t)$. Along with the initial conditions $x(0)$ and $dx/dt|_{t=0}$ it determines the horizontal motion of the car, i.e. $x(t)$. The vertical motion is determined by the shape of the rollercoaster, $y(x)$, as long as the car stays on the track. The car *will fly off the track* if and when the normal reaction force becomes zero:

$$R = 0. \tag{4.18}$$

Indeed, this equation says that the car and the track no longer act on one another, which means that the car is free to leave the track.

Therefore, as we monitor the r.h.s. of (4.17) using the value of $(dx/dt)^2$ obtained from (4.9*b*) and find that at some moment, it vanishes, then this is when and where the car flies off. Obviously (from (4.17)), this can only occur when $y'' < 0$, i.e. on a concave-down section (a bump) of the track.

Once the car has flown off, its motion is described by two simple equations, usually presented in Calculus I and III and in an elementary Physics course:

$$x(t) = x_l + (v_l)_x(t - t_l) \tag{4.18a}$$

$$y(t) = y_l + (v_l)_y(t - t_l) - \frac{g(t-t_l)^2}{2}, \tag{4.18b}$$

where t_l is the time the car leaves the track, (x_l, y_l) is the location on the track where it does so, and

$$(v_l)_x = \left.\frac{dx}{dt}\right|_{t=t_l}, \tag{4.19a}$$

$$(v_l)_y = \left.\frac{dy}{dt}\right|_{t=t_l} = \left(\frac{dy}{dx}\right)\cdot\left.\frac{dx}{dt}\right|_{t=t_l} \equiv y'\cdot(v_l)_x \tag{4.19b}$$

are the x- and y-components of the velocity at this moment.

Let us note that Eq. (4.10*b*) for a general form of the track (i.e., for a general $y(x)$) cannot be solved analytically. Nevertheless, it can be easily solved numerically, e.g., in Matlab.

In the homework, you will explore three modifications of the problem considered above:
(i) when the shape of the track is defined by parametric equations rather than as $y(x)$;
(ii) when the car moves on the bottom side of the track; and
(iii) when the friction between the car and the surface of the track is included in the model.