5 Markov Chain Models

In this lecture we will pursue two goals. To introduce the first of them, we need a brief preamble.

Suppose a system at any moment can occupy a finite number of states. (E.g., the weather at a given city on a summer day may be sunny, cloudy, or rainy.) Suppose the system moves from one state to another with some probabilities. If these ***transition probabilities*** depend only on the *current* state of the system (and not on which previous states the system has visited), then the evolution of the system from one state to the next is called a ***Markov process***, or a ***Markov chain***.

Our *first goal* will be to practice writing balance equations for Markov Chain models. The idea of these balance equations is, roughly speaking, this:

the inflow into a given state of the system
must equal
the outflow from this state.

What flows in and out will be specified for each given model. Using the balance equations, we will determine an equilibrium distribution of states of the system. (E.g., in the weather example above, we would determine how many sunny, cloudy, and rainy days the given city has in an average summer).

Our *second goal* will be to review the role of eigenvectors in the description of physical systems. Namely, we will review how eigenvectors are related to the limiting distribution of states reached by the system at large times.

Before we begin with an introductory example, let us first list some basic facts from Linear Algebra that we will require. Unless otherwise stated, below $\underline{x}$ is an M-dimensional vector:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}$$

and A is an $M \times M$ matrix. Also, I is the $M \times M$ identity (unity) matrix.

Fact 1 If A is written as a collection of its columns:

$$A = [\underline{A}_1, \underline{A}_2, \ldots \underline{A}_M],$$

then

$$A\underline{x} = x_1\underline{A}_1 + x_2\underline{A}_2 + \ldots + x_M\underline{A}_M, \qquad (5.1)$$

where $x_1, \ldots, x_M$ are defined above. Equation (5.1) will be used *extensively* in what follows.

Fact 2a Matrix A is ***singular*** if and only if there is $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$. If A is not singular, it is called ***nonsingular***.

Fact 2b Linear system

$$A\underline{x} = \underline{b}$$

has a *unique* solution $\underline{x}$ for *any* $\underline{b}$ if and only if A is nonsingular.

Fact 3 If $(A - \lambda I)$ is singular, then the scalar λ is called an ***eigenvalue*** of A. Then (see Fact 2a) there is a vector $\underline{v} \neq \underline{0}$ such that $(A - \lambda I)\underline{v} = \underline{0}$, or

$$A\underline{v} = \lambda\underline{v}.$$

This $\underline{v}$ is called the ***eigenvector*** of A corresponding to the eigenvalue λ .

Example 1 By reviewing its donation record, the alumni office of a college finds that 80% of its alumni who contribute to the annual fund one year will also contribute the next year, and 30% of those who do not contribute one year will contribute the next. How will the donation record evolve over the years? In particular, what will it be in 12 years for a newly graduating class in which no alum donated immediately after the graduation? (Assume the class has 1000 graduates).

Solution: Let us denote the number of alumni who donated in the k-th year by $D^{(k)}$ and the number of those who did not donate in that year, by $N^{(k)}$. Then:

$$\underbrace{D^{(k+1)}}_{\substack{\text{number of alumni} \\ \text{donating in year } k+1}} = \underbrace{0.8 \cdot D^{(k)}}_{\substack{80\% \text{ of those who} \\ \text{donated in year } k}} + \underbrace{0.3 \cdot N^{(k)}}_{\substack{30\% \text{ of those who did} \\ \text{not donate in year } k}}$$

Similarly,

$$N^{(k+1)} = 0.2 \cdot D^{(k)} + 0.7 \cdot N^{(k)}.$$

We can write these two equations in matrix form:

$$\begin{pmatrix} D \\ N \end{pmatrix}^{(k+1)} = P \cdot \begin{pmatrix} D \\ N \end{pmatrix}^{(k)}$$

where

$$P = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}.$$

Note that since the total number of alumni is assumed to be the same, we can divide both sides of the above equation by $D^{(0)} + N^{(0)} = D^{(k)} + N^{(k)} = 1000$. If we denote

$$d^{(k)} = \frac{D^{(k)}}{D^{(0)} + N^{(0)}}, \qquad n^{(k)} = \frac{N^{(k)}}{D^{(0)} + N^{(0)}}$$

to be the relative numbers of donating and non-donating alumni, we rewrite the above equation as

$$\begin{pmatrix} d \\ n \end{pmatrix}^{(k+1)} = P \begin{pmatrix} d \\ n \end{pmatrix}^{(k)} . \tag{5.2}$$

Let us call $\underline{x}^{(k)} = \begin{pmatrix} d \\ n \end{pmatrix}^{(k)}$ the ***state vector*** of the Markov process described by Eq. (5.2). If we further denote

$$\underline{x}^{(k)} \equiv \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix},$$

then obviously,

$$\sum_{i=1}^{2} x_i^{(k)} = 1 \qquad \text{for all } k. \tag{5.3}$$

(This expresses the conservation of the number of alumni.) Let us also refer to $\underline{x}^{(0)}$ as the ***initial state vector*** and to P as the ***transition matrix*** of the Markov process. Note that the entries of the transition matrix satisfy the condition

$$\sum_{i=1}^{2} p_{ij} = 1 \qquad \text{for } j = 1, 2, \tag{5.4}$$

i.e. the *column sums* of P equal one. Indeed:

$p_{11}(=0.8)$ is the probability that the system will "move" from state 1 to state 1 (donating),

$p_{21}(=0.2)$ is the probability that the system will move from state 1 to state 2 (non-donating).

Since there are no other possibilities (i.e., no other states), then $p_{11}+p_{21}$ must equal 1. In words, this means that an alumnus who donated in year k, will either donate or not donate in the next year. Similarly, one can conclude that $p_{12}+p_{22}=1$. (To practice, say what this equation means in words.)

Returning to the solution of our example and using the notation $\underline{x}^{(k)}$ for $\begin{pmatrix} d \\ n \end{pmatrix}^{(k)}$, we rewrite Eq. (5.2) as:

$$\underline{x}^{(k+1)} = P\underline{x}^{(k)}. \tag{5.5}$$

This allows one to determine $\underline{x}^{(k)}$ recursively. One can also write a formula relating $\underline{x}^{(k)}$ directly to $\underline{x}^{(0)}$:

$$\begin{aligned} \underline{x}^{(k+1)} &= P \cdot P \cdot \underline{x}^{(k-1)} = \\ &= P \cdot P \cdot P \cdot \underline{x}^{(k-2)} = \dots \\ &= P^{k+1} \cdot \underline{x}^{(0)}. \end{aligned}$$

Thus:

$$\underline{x}^{(k)} = P^k \underline{x}^{(0)}. \tag{5.6}$$

Using either (5.5) or (5.6) , we can find $\underline{x}^{(k)}$ starting with $\underline{x}^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which is given as the initial state vector in this example:

k	1	2	3	4	5	6	7	8	9	10	11	12
$x_1^{(k)}$	0.3	0.45	0.525	0.563	0.581	0.591	0.595	0.598	0.599	0.599	0.600	0.600
$x_2^{(k)}$	0.7	0.55	0.475	0.437	0.419	0.409	0.405	0.402	0.401	0.401	0.400	0.400

Thus, after a few years, the state vector converges to a fixed vector. In this limit, about 60% of alumni will contribute, and 40% will not contribute, to the annual fund.

This limiting behavior can be understood from the eigenvalues and eigenvectors of matrix P, as explained below. First, let us find these eigenvalues and eigenvectors. We find the eigenvalues using the characteristic equation for matrix P:

$$\begin{aligned} \det(P-\lambda I)=0 \quad &\Rightarrow \quad \begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = 0 \\ &\Rightarrow \quad \lambda^2 - 1.5\lambda + 0.5 = 0 \\ &\Rightarrow \quad (\lambda-1)(\lambda-0.5)=0 \\ &\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 0.5. \end{aligned}$$

Now let us find the eigenvectors. For $\underline{\lambda = \lambda_1}$:

$$P\underline{v} = 1\cdot\underline{v} \quad \Rightarrow \quad \begin{pmatrix} 0.8-1 & 0.3 \\ 0.2 & 0.7-1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} s,$$

where s is an arbitrary constant (recall from Linear Algebra that an eigenvector is defined up to an arbitrary nonzero factor). In the application we are considering, the components of the state vectors satisfy condition (5.3). Therefore, it is convenient to select s so that the components of $\underline{v_1}$ satisfy the same condition. Thus we take

$$\underline{v_1} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}. \tag{5.7a}$$

For $\underline{\lambda = \lambda_2}$:

$$P\underline{v} = 0.5\underline{v} \quad \Rightarrow \quad \begin{pmatrix} 0.8-0.5 & 0.3 \\ 0.2 & 0.7-0.5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} s.$$

Since here we cannot satisfy condition (5.3), we simply take $s = 1$. Thus,

$$\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5.7b}$$

Next, since $\underline{v_1}$ and $\underline{v_2}$ are linearly independent, they form a basis in R^2, and hence any initial state vector can be represented as their linear combination:

$$\underline{x}^{(0)} = c_1\underline{v_1} + c_2\underline{v_2}, \tag{5.8}$$

for some constants c_1, c_2. Then:

$$\begin{aligned} \underline{x}^{(1)} = P\underline{x}^{(0)} &= Pc_1\underline{v_1} + Pc_2\underline{v_2} \\ &= c_1P\underline{v_1} + c_2P\underline{v_2} \\ &= c_1\lambda_1\underline{v_1} + c_2\lambda_2\underline{v_2}, \end{aligned}$$

$$\begin{aligned} \underline{x}^{(2)} = P\underline{x}^{(1)} &= Pc_1\lambda_1\underline{v_1} + Pc_2\lambda_2\underline{v_2} \\ &= c_1\lambda_1^2\underline{v_1} + c_2\lambda_2^2\underline{v_2}, \end{aligned}$$

$$\dots$$

$$\underline{x}^{(k)} = P^k\underline{x}^{(0)} = c_1\lambda_1^k\underline{v_1} + c_2\lambda_2^k\underline{v_2}. \tag{5.9}$$

Now, since $\lim_{k\to\infty} \lambda_1^k = \lim_{k\to\infty} 1^k = 1$ and $\lim_{k\to\infty} \lambda_2^k = \lim_{k\to\infty} 0.5^k = 0$, we finally obtain:

$$\lim_{k\to\infty} \underline{x}^{(k)} = c_1\underline{v_1}.$$

Since both $\underline{x}^{(k)}$ and $\underline{v_1}$ satisfy condition (5.3) ($\underline{x}^{(k)}$ — by its meaning and $\underline{v_1}$ — by design), then $c_1 = 1$. Thus

$$\lim_{k\to\infty} \underline{x}^{(k)} = \underline{v_1} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}, \tag{5.10}$$

which agrees with the result shown in the table after Eq. (5.6).

We have completely solved the problem. However, we can still obtain additional information from it. To this end, let us compute $\lim_{k\to\infty} \underline{x}^{(k)}$ using (5.6) rather than (5.5) (as we did above). Since, as we showed, $\underline{x}^{(k)}$ tends to a limit, then by (5.6), so does P^k. Let

$$\lim_{k\to\infty} P^k \equiv Q \equiv [\underline{q_1}, \underline{q_2}]. \tag{5.11}$$

Then from (5.6) and (5.10),

$$Q\underline{x}^{(0)} = \underline{v_1} \tag{5.12}$$

for any $\underline{x}^{(0)} \neq \underline{v_2}$ (in the latter case, $c_1 = 0$; we will return to it later).

To obtain an explicit form of the 2×2 matrix Q, let us act with it on some two linearly independent vectors. As a first such vector, let us take $\underline{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then from (5.11) and (5.1),

$$\underline{q_1} + \underline{0} = \underline{v_1}. \tag{5.13a}$$

Now, let us take $\underline{x}^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and similarly obtain:

$$\underline{0} + \underline{q_2} = \underline{v_1}. \tag{5.13b}$$

These equations imply that

$$Q = [\underline{v_1}, \underline{v_1}]. \tag{5.14}$$

This can be verified by computing Q directly from (5.11), by looking at higher and higher powers k of P.

We can proceed to obtain even more information. Let now $\underline{x}^{(0)} = \underline{v_2}$ (ignoring for the moment the fact that $\underline{v_2}$ does not satisfy condition (5.3)). Then in (5.9), $c_1 = 0, c_2 = 1$. From the equation stated before Eq. (5.10), we obtain:

$$Q\underline{v_2} = \underline{0}.$$

Let $\underline{v_2} = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$. Then

$$[\underline{v_1}, \underline{v_1}] \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \underline{0},$$

and using (5.1),

$$\underline{v_1} \cdot v_{12} + \underline{v_1} \cdot v_{22} = \underline{0} \quad \Rightarrow \quad v_{12} = -v_{22}.$$

Of course, we know this from the explicit calculation, Eq. (5.7b). However, the same method can be applied to other problems where we do not even need to know P explicitly. Thus, let us write this result as:

$$\sum_{i=1}^{2} v_{i2} = 0. \tag{5.15}$$

Let us now consider another example to reinforce the techniques used and the conclusions obtained.

Example 2 A car rental agency has three rental locations, which we label as locations $1, 2,$ and 3. A customer may rent a car from any of the three locations and return it to any of them. The manager finds that customers return the cars to various locations according to the following probabilities:

If rented from 1, then return to 1, 2, 3 with probabilities 80%, 10%, 10%.
If rented from 2, then return to 1, 2, 3 with probabilities 30%, 20%, 50%.
If rented from 3, then return to 1, 2, 3 with probabilities 20%, 60%, 20%.

Suppose that initially, the company's total car fleet of 1000 cars is at location 2. How will the number of cars at all three locations evolve over time? (Assume that the record is taken every week.)

Solution: Let $x_1^{(k)}, x_2^{(k)}, x_3^{(k)}$ be the numbers of cars at the three locations at the k-th recording instance. Then at the $(k+1)$-st recording instance:

$$\underbrace{x_1^{(k+1)}}_{\substack{\text{number of cars at}\\ \text{location 1 at the}\\ (k+1)\text{-th instance}}} = \underbrace{0.8x_1^{(k)}}_{\substack{\text{number of cars taken}\\ \text{from 1 and}\\ \text{returned to 1}}} + \underbrace{0.3x_2^{(k)}}_{\substack{\text{number of cars taken}\\ \text{from 2 and}\\ \text{returned to 1}}} + \underbrace{0.2x_3^{(k)}}_{\substack{\text{number of cars taken}\\ \text{from 3 and}\\ \text{returned to 1}}}$$

Similarly,

$$x_2^{(k+1)} = 0.1x_1^{(k)} + 0.2x_2^{(k)} + 0.6x_3^{(k)},$$
$$x_3^{(k+1)} = 0.1x_1^{(k)} + 0.5x_2^{(k)} + 0.2x_3^{(k)}.$$

In matrix form:

$$\underline{x}^{(k+1)} = P\underline{x}^{(k)},$$

where

$$P = \begin{pmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.6 \\ 0.1 & 0.5 & 0.2 \end{pmatrix}.$$

As before,

$$\sum_{i=1}^{3} p_{ij} = 1 \quad \text{for } j = 1, 2, 3.$$

We now list $\underline{x}^{(k)}$, assuming $\underline{x}^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Here, as previously in Example 1, we normalize the numbers of cars at each location to the total number of cars, so that a condition analogous to (5.3) holds.

k	1	2	3	4	5	6	7	8	9	10	11	12
$x_1^{(k)}$	0.3	0.40	0.477	0.511	0.533	0.544	0.550	0.553	0.555	0.556	0.557	0.557
$x_2^{(k)}$	0.2	0.37	0.252	0.261	0.240	0.238	0.233	0.232	0.231	0.230	0.230	0.230
$x_3^{(k)}$	0.5	0.23	0.271	0.228	0.227	0.219	0.217	0.215	0.214	0.214	0.213	0.213

Thus, again, as in Example 1, the state vector converged to a fixed vector after a few weeks. On average, there will be 557, 230, and 213 cars at locations 1, 2, 3.

Let us re-obtain this result via the eigenvector–eigenvalue analysis. It is possible to follow the method used to obtain Eqs. (5.7) in Example 1. However, a much easier approach is to use Matlab. The command is

```
>> [V,D]= eig(P)
```

(type `help eig` for the explanation). Its result is:

$$\underline{\lambda_1 = 1} \qquad \underline{\lambda_2 = 0.547} \qquad \underline{\lambda_3 = -0.347}$$

$$\underline{v_1} \simeq \begin{pmatrix} 0.557 \\ 0.230 \\ 0.213 \end{pmatrix} \qquad \underline{v_2} \simeq \begin{pmatrix} 0.816 \\ -0.431 \\ -0.385 \end{pmatrix} \qquad \underline{v_3} \simeq \begin{pmatrix} 0.078 \\ -0.743 \\ 0.665 \end{pmatrix}.$$

Note that $\underline{v_1}$ satisfies a condition analogous to (5.3) (by design) and $\underline{v_2}$, $\underline{v_3}$ satisfy a condition analogous to (5.15), with $\sum_{i=1}^{2}$ being replaced by $\sum_{i=1}^{3}$. The counterparts of (5.8) and (5.9) are:

$$\underline{x}^{(0)} = c_1\underline{v_1} + c_2\underline{v_2} + c_3\underline{v_3},$$

$$\underline{x}^{(k)} = c_1\lambda_1^k\underline{v_1} + c_2\lambda_2^k\underline{v_2} + c_3\lambda_3^k\underline{v_3}.$$

Since $\lambda_1 = 1$ and $|\lambda_2|, |\lambda_3| < 1$, then

$$\lim_{k\to\infty} \underline{x}^{(k)} = c_1\underline{v_1}.$$

As before, since $\underline{v_1}$ satisfies a counterpart of (5.3), then $c_1 = 1$ (for *any* $\underline{x}^{(0)}$ that also satisfies a counterpart of (5.3)), so that

$$\lim_{k\to\infty} \underline{x}^{(k)} = \underline{v_1} = \begin{pmatrix} 0.557 \\ 0.230 \\ 0.213 \end{pmatrix},$$

which agrees with the result found in the table on the previous page.

Next, in analogy with (5.11), let

$$\lim_{k\to\infty} P^k = Q \equiv [\underline{q_1}, \underline{q_2}, \underline{q_3}]$$

and then obtain the explicit form of $\underline{q_1}, \underline{q_2}, \underline{q_3}$. Similarly to (5.12), one has:

$$Q\underline{x}^{(0)} = \underline{v_1}$$

for any $\underline{x}^{(0)}$ for which $c_1 \neq 0$ (i.e. when $\underline{x}^{(0)}$ does not coincide with either $\underline{v_2}$ or $\underline{v_3}$ or their linear combination). Take $\underline{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then similarly to (5.13a),

$$\underline{q_1} + \underline{0} + \underline{0} = \underline{v_1}.$$

Taking $\underline{x}^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ yields, respectively,

$$\underline{0} + \underline{q_2} + \underline{0} = v_1 \qquad \text{and} \qquad \underline{0} + \underline{0} + \underline{q_3} = \underline{v_1}.$$

Therefore, $\underline{q_1} = \underline{q_2} = \underline{q_3} = \underline{v_1}$, whence

$$Q = [\underline{v_1}, \underline{v_1}, \underline{v_1}].$$

Finally, take $\underline{x}^{(0)} = \underline{v_2}$; then (since $c_1 = 0$)

$$Q\underline{v_2} = \underline{0}.$$

Denoting $\underline{v_2} = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}$ and using the above expression for Q, we find:

$$\underline{v_1} \cdot v_{12} + \underline{v_1} \cdot v_{22} + \underline{v_1} \cdot v_{32} = 0,$$

whence

$$v_{12} + v_{22} + v_{32} = 0,$$

or

$$\sum_{i=1}^{3} v_{i2} = 0.$$

This is similar to (5.15). By the same token,

$$\sum_{i=1}^{3} v_{i3} = 0.$$

From the calculations we have done so far, we can draw the following conclusions.

1. Transition matrices (i.e. matrices with $\sum_{i=1}^{M} p_{ij} = 1, \quad j = 1, ...M$; see Eq. (5.4)) have a unique eigenvector $\underline{v_1}$ with $\lambda_1 = 1$ and such that $v_{i1} \geq 0$.

2. The other eigenvalues, λ_2, λ_3... of the transition matrices are all less than 1 in magnitude. Moreover, their eigenvectors satisfy: $\sum_{i=1}^{M} v_{ij} = 0, \quad j = 2, ...M$; see Eq. (5.15).

Question: Do these observations hold for all transition matrices?

Answer: No.

For example, $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a transition matrix. But $\lambda_1 = 1, \lambda_2 = -1$ (which is not less than 1 in magnitude) and $\underline{v_1} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since

$$\lim_{k\to\infty} |\lambda_2^k| \neq 0,$$

then $\lim_{k\to\infty} \underline{x}^{(k)}$ no longer exist, and neither does $\lim_{k\to\infty} P^k$ (indeed, $P^2 = I,\ P^3 = P,\ P^4 = I$, etc).

However, the conclusions stated above do hold for a *restricted* class of so called ***regular*** transition matrices.

Definition A transition matrix is called ***regular***[7] if *some* integer power of it has all positive

[7] or, sometimes, *stochastic*

entries.

For regular transition matrices, the following theorem is valid.

Theorem If P is a regular transition matrix, then as $k \to \infty$:

1. $$P^k \to [\underline{v_1}, \underline{v_1}, \ldots \underline{v_1}],$$

   where $\underline{v_1}$ is the eigenvector of P corresponding to the eigenvalue $\lambda_1 = 1$.

2. This eigenvector is unique and all of its entries are positive; then they can be normalized so as to have
   $$\sum_{i=1}^{M} v_{i1} = 1. \qquad (5.16a)$$

3. The other eigenvalues of P are less than one in magnitude and their eigenvectors satisfy
   $$\sum_{i=1}^{M} v_{ij} = 0, \qquad j = 2, \ldots M. \qquad (5.16b)$$

4. (This conclusion follows from item 2 and the first half of item 3.) As $k \to \infty$,
   $$P^k \underline{x}^{(0)} \to \underline{v_1}$$
   for any *state vector* $\underline{x}^{(0)}$ (recall that a state vector must satisfy (5.16a)).

The most difficult statements to prove is that there is only one eigenvalue $\lambda_1 = 1$ and that the corresponding eigenvector has all positive entries. These statements are usually proved in advanced courses on Linear Algebra. While they may seem to be of purely academic interest, it is not the case. In fact, these statements have important applications in the theory of ranking algorithms and, in particular, in the theory of PageRank algorithm, used by Google in its early years. They also find applications in the theory of wireless signal transmission and reception, where multiple transmitters and receivers are considered (as it is common in practice).

The fact that there is *an* eigenvalue $\lambda_1 = 1$ (not necessarily unique) is easier to prove; this is left as a homework problem.

As yet another piece of notation, we note that $\underline{v_1}$, defined in item 4 of the Theorem, is called the ***steady state vector*** of a regular (see the Definition above) Markov chain. This new name makes sense for a number of reasons. First, as the definition in item 4 above — which is just Eq. (5.6) in the limit $k \to \infty$ — says, $\underline{v_1}$ is approached as a *steady*, i.e. non-changing, state of the system in the limit of many iterations. Second, this fact can also be viewed from the perspective of Eq. (5.5) rather than from that of Eq. (5.6). Indeed, if in (5.5) we let $k \to \infty$ and note that $\infty + 1 = \infty$, we obtain *the equation satisfied by the steady state vector*:

$$P\,\underline{x}^{(\infty)} = \underline{x}^{(\infty)}.$$

This is the same equation as satisfied by the eigenvector of P with eigenvalue $\lambda = 1$:

$$P\,\underline{v_1} = 1 \cdot \underline{v_1}.$$

We will now consider a different application where the steady state vector of a Markov chain appears.

Example 3 This example introduces, in a very simple setting, an application of Linear Algebra to interrelations between prices and outputs in an economic system. The ideas behind the theory developed below belong to Wassily Leontief, a Russian economist who emigrated, soon after earning his M.A. degree from the Leningrad State University in 1924, to Germany and then to the United States. A series of works where a comprehensive theory of an economic system was developed by Leontief, appeared in the 1930–1940's (when he was in the US) and won him the Nobel Prize in economics in 1973.

The ***closed***, or input-output, model considered in this example is closely related (as far as its mathematics is concerned) to the models considered in Examples 1 and 2.

Three homeowners — a carpenter, an electrician, and a plumber — mutually agree to make repairs in their three homes. They agree to work a total of ten days each according to the following schedule:

The carpenter spends 2 days in his own home, 4 days in the electrician's home, and 4 days in the plumber's home.

The electrician spends 1 day in the carpenter's home, 5 days in his own home, and 4 days in the plumber's home.

The plumber spends 6 days in the carpenter's home, 1 day in the electrician's home, and 3 days in his own home.

For tax purposes, they must report and pay each other a reasonable daily wage, even for the work each does on his own home. They agree to adjust their daily wages so that each homeowner will come out even (i.e., the total amount paid out by each is the same as the total amount each receives).

What daily wages should they charge?

Solution: Let p_1, p_2, p_3 be the daily wages of the carpenter, electrician, and plumber. To satisfy the ***equilibrium condition*** that each homeowner comes out even, we require that for each of them,

$$\text{total expenditures} = \text{total income.} \qquad (*)$$

To write out equations corresponding to condition $(*)$, let us first present the data given in the problem in the form of a table:

	Days worked by		
Days worked in the house of	C	E	P
C	2	1	6
E	4	5	1
P	4	4	3

For the carpenter, $(*)$ yields:

$$\underbrace{2p_1}_{\text{C pays C}} + \underbrace{p_2}_{\text{C pays E}} + \underbrace{6p_3}_{\text{C pays P}} = \underbrace{10p_1.}_{\text{C receives}}$$

(What is the significance of the number '10' here?) Similarly, for the electrician and the plumber:

$$4p_1 + 5p_2 + p_3 = 10p_2,$$

$$4p_1 + 4p_2 + 3p_3 = 10p_3.$$

Note that the l.h.s. of these equations are just the rows of the table above.

Dividing these equations by ten (i.e., by the total number of days), we write them in matrix form as:

$$\mathcal{E}\underline{p} = \underline{p}, \tag{5.17}$$

where $\underline{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ is the price vector, charged by each person for the unit of his time (or, the unit of output). In (5.17), $\mathcal{E} = (e_{ij})$ is the ***exchange***, or the ***input-output***, matrix, whose entries e_{ij} repeat the entries of the table above divided by 10. In general, e_{ij} is the fraction of the total output of person j used by person i.

You may recognize Eq. (5.17) as being the eigenvalue equation with $\lambda = 1$. Then, the theorem stated before this Example guarantees that there is a unique solution of (5.17) with all positive entries (i.e., no one performs his work free of charge) as long as the exchange matrix is regular according to the Definition stated before Eqs. (5.16). That is:

1) *All* entries of $\mathcal{E}$ or of some of its powers, $\mathcal{E}^m$, are positive; and

2) $\sum_{i=1}^{M} e_{ij} = 1, \quad j = 1, ...M$ (in our example, $M = 3$).

Let us note that condition 1) means that there is an exchange relationship, either direct or indirect (i.e., via others) between any two persons in the model.

In our example, matrix $\mathcal{E}$ meets both conditions (verify!), and hence a unique price vector exists. It can be found either by hand or by Matlab (as a decimal) or by Mathematica (exactly). The result is:

$$\underline{p} = \begin{pmatrix} 31 \\ 32 \\ 36 \end{pmatrix} \cdot s,$$

where s is an arbitrary number. Taking s to be, say, 12, we obtain reasonable daily wages for the year 2018.

In this example we have seen that the eigenvalue equation playing the key role in the theory of Markov Chain models, also occurs in a very different context of a closed economic model. As we have mentioned earlier, the same equation also occurs in the theory of Google's PageRank algorithm.

Our last example will be related to Example 3, but not to Examples 1, 2, and the project paper. We will present it just to give a glimpse at a more practically relevant theory than the one found in Example 3. We will also use this example to practice setting up balance equations.

In the *closed* economic model considered above, the outputs of all producers are distributed among them, and these outputs are fixed. (E.g., the numbers of days each person in Example 3 worked, was fixed.) The goal was to determine the prices for these outputs so that the equilibrium condition $(*)$ of Example 3 be satisfied.

In the ***open*** model, which we will consider below, the producers attempt to satisfy not only each other's demand but also an outside demand for their outputs. The goal is now to determine levels of the outputs that the producers need to maintain in order to satisfy an outside demand. What is considered known in this model is how much the unit of one producer's output depends on the outputs of the others. We will quantify this below.

Example 4 Let us suppose that the carpenter, electrician, and plumber founded a firm that provides services to their town. The three people then determined that to produce \$1 worth of their individual outputs, they require the following amounts of the outputs of the other two people:

\$1 of C requires:	\$0.10 of C,	\$0.35 of E,	\$0.05 of P;
\$1 of E requires:	\$0.06 of C,	\$0.09 of E,	\$0 of P;
\$1 of P requires:	\$0.04 of C,	\$0.11 of E,	\$0.07 of P.

Suppose the carpenter received an order from the town hall for \$1000. How much output (measured in \$\$) must each of the three persons in the firm produce to fulfill this order?

Solution: Let x_1, x_2, x_3 be the outputs, measured in dollars, of the carpenter, electrician, and plumber.
For the carpenter, we have the following balance equation:

$$\underbrace{x_1}_{\substack{\text{dollar output}\\ \text{of } C}} = \underbrace{0.10\cdot x_1}_{\substack{\text{part of } C\text{'s}\\ \text{output needed}\\ \text{to support}\\ \text{his own}\\ \text{functioning}}} + \underbrace{0.06x_2}_{\substack{\text{part of } C\text{'s}\\ \text{output needed}\\ \text{to support}\\ E\text{'s}\\ \text{functioning}}} + \underbrace{0.04x_3}_{\substack{\text{part of } C\text{'s}\\ \text{output needed}\\ \text{to support}\\ P\text{'s}\\ \text{functioning}}} + \underbrace{1000}_{\substack{\text{the outside}\\ \text{demand for}\\ C\text{'s output}}}$$

Similarly, for the electrician and plumber:

$$x_2 = 0.35x_1 + 0.09x_2 + 0.11x_3 + 0,$$

$$x_3 = 0.05x_1 + 0\cdot x_2 + 0.07x_3 + 0.$$

In matrix form:

$$\underline{x} - C\underline{x} = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}, \qquad (5.18)$$

where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the output vector and

$$C = \begin{pmatrix} 0.10 & 0.06 & 0.04 \\ 0.35 & 0.09 & 0.11 \\ 0.05 & 0 & 0.07 \end{pmatrix}$$

is called the ***consumption matrix***. A consumption matrix is called ***productive*** if a solution of (5.18) with *all positive entries* exists. It can be shown that a consumption matrix is productive if either all its column sums or all its row sums are less than 1. This is clearly the case in this example. An easy calculation (e.g., with Matlab) yields:

$$x_1 \approx \$1144, \qquad x_2 \approx \$447, \qquad x_3 \approx \$62.$$