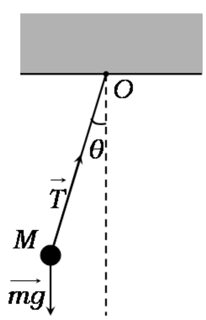
6 Simple pendulum. Part I: The Basics

In this lecture we will pursue two main goals. First, using the simple (a.k.a. mathematical) pendulum as an example, we will introduce a model that is of fundamental importance to applied mathematics and physics and arises in a great number of very different applications. This model is called the *harmonic oscillator* model. We will also show, at the end of the lecture and in a homework problem, how the same mathematical model arises in different physical applications.

Second, we will introduce a mathematical description for a *real-valued* solution of this model based on *complex numbers*. As the harmonic oscillator model arises everywhere in applied mathematics, so does the aforementioned complex representation of real-valued solutions.

In this lecture, and in the homework, we will also see more applications of the Taylor expansion and linearization.

6.1 Derivation of the equation of the model



Consider mass m, located at point M, and suspended from point O on a rod OM. Our first step will be to derive equations of the motion for this simple system.

It is possible to derive these equations using the x- and y-coordinates of m. However, both these equations and the calculations leading to them will be ugly. This occurs because the natural degrees of freedom of point M are *not* x and y but rather the angle θ between OM and the vertical line.

To derive equations of motion for this angle, one needs to use the Second Newton's Law for rotational motion, which states:

$$I\frac{d^2\vec{\theta}}{dt^2} = \sum_i \vec{r_i} \times \vec{F_i}\,. \tag{6.1}$$

On the l.h.s. of this equation, I is the moment of inertia of the object (see below) and $\vec{\theta}$ is the vector whose magnitude equals the angle of rotation and whose direction is along the rotation axis. More specifically, the direction of $\vec{\theta}$ follows the right-hand rule. For example, in the picture above, $\vec{\theta}$ is pointing *into* the page.[8] On the r.h.s. of (6.1), $\vec{F_i}$ is a force and $\vec{r_i}$ is the radius-vector pointing from the axis of rotation to the point where $\vec{F_i}$ is applied, the summation is over all forces acting on the object, and '$\times$' denotes the cross product. In our case, (6.1) reduces to:

$$mr^2\,\frac{d^2\vec{\theta}}{dt^2} = \vec{r}\times m\vec{g} + \vec{r}\times\vec{T}\,,$$

where the forces $m\vec{g}$ and $\vec{T}$ are marked in the figure above and $\vec{r} = \vec{OM}$. Since $\vec{r}$ and $\vec{T}$ are parallel, then $\vec{r}\times\vec{T} = \vec{0}$. Now, it is easy to verify (do it!) that in the case depicted in this

[8]Explanation: Imagine a right-threaded screw that is being rotated from the dashed line (where $\theta = 0$) towards the position of the pendulum shown in the picture. This screw will go *into* the page.

figure, $\vec{r} \times m\vec{g}$ points *out* of the page, while $\vec{\theta}$ points *into* the page (as we have noted earlier). Thus, $\vec{\theta}$ and $\vec{r} \times m\vec{g}$ are aligned along the same axis (the axis of rotation), but are oppositely directed. Therefore, we can rewrite (6.1) in the scalar form as:

$$mr^2 \frac{d^2\theta}{dt^2} = -r \cdot mg \cdot \sin\theta \, .$$

Denoting $r \equiv l$, the length of the pendlum, we finally write this as:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta \, . \tag{6.2}$$

6.2 Nondimensionalization

The standard step, as you have seen earlier in Lecture 3, is to nondimensionalize the equation of motion. To see what change of variables we need for this, we rewrite (6.2) as

$$\frac{1}{(g/l)} \frac{d^2\theta}{dt^2} = -\sin\theta \, .$$

This suggests that we take

$$\tau = \sqrt{\frac{g}{l}} t \, .$$

Verify that the τ so defined is nondimensional. Next, as we did in Lectures 2 – 4, we use the Chain Rule to obtain:

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \sqrt{\frac{g}{l}} \frac{d}{d\tau} \, ,$$

$$\frac{d^2}{dt^2} = \sqrt{\frac{g}{l}} \frac{d}{d\tau} \left(\sqrt{\frac{g}{l}} \frac{d}{d\tau} \right) = \frac{g}{l} \frac{d^2}{d\tau^2} \, ,$$

where at the last step we have used the fact that $(g/l) = \text{const}$. Substituting this into (6.2) yields

$$\frac{d^2\theta}{d\tau^2} = -\sin\theta$$

or, using the overdot notation for $\frac{d}{d\tau}$, as in Lecture 3:

$$\ddot{\theta} = -\sin\theta \, . \tag{6.3}$$

This is the nondimensional equation of a pendulum.

6.3 Linearization of (6.3) and the harmonic oscillator model

Another standard step followed by scientists who are to analyze a nonlinear equation of motion (like (6.3)) is to first focus on the vicinity of the equilibrium position(s) of the system. The simple pendulum has two equilibrium positions, where $\ddot{\theta} = 0$: $\theta_E = 0$ and $\theta_E = \pi$. Therefore, to study the motion of the pendulum, we linearize Eq. (6.3) near these equilibria. For most of this lecture, we will consider the vicinity of the stable equilibrium, $\theta_E = 0$, and will only briefly consider the vicinity of the unstable equilibrium, $\theta_E = \pi$, at the end of the lecture.

So, when $\theta \ll 1$, one has the Taylor (more exactly, Maclaurin) expansion:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots \, .$$

(Note that everywhere in this section, θ is in radians, *not* in degrees.) Keeping only the first term of the expansion, i.e. $\sin\theta \approx \theta$ (recall that $\lim_{\theta\to 0}(\sin\theta/\theta) = 1$), one obtains from (6.3):

$$\ddot{\theta} = -\theta\,, \qquad \text{or} \qquad \ddot{\theta} + \theta = 0\,. \tag{6.4}$$

Equation (6.4) is called the harmonic oscillator model and arises in a great many applications.

6.4 Real form of the solution of (6.4); Linear superposition principle

As you know (and can easily verify), both $\theta = \cos\tau$ and $\theta = \sin\tau$ solve (6.4). In fact,

$$\theta = c_1\cos\tau + c_2\sin\tau\,, \tag{6.5}$$

where c_1, c_2 are arbitrary constants, solves (6.4). Indeed, substituting (6.5) into (6.4), one obtains:

$$(c_1\cos\tau)^{\cdot\cdot} + (c_2\sin\tau)^{\cdot\cdot} = -(c_1\cos\tau) - (c_2\sin\tau),$$

or

$$c_1(\cos\tau)^{\cdot\cdot} + c_2(\sin\tau)^{\cdot\cdot} = -c_1\cos\tau - c_2\sin\tau,$$

where we moved c_1, c_2 outside of $(d^2/d\tau^2)$ because they are constant. Now, since each of $\sin\tau$ and $\cos\tau$ satisfies (6.4), then the first term on the l.h.s. cancels with the first term on the r.h.s., and similarly, the second terms also cancel out. This proves that (6.5) is the solution of (6.4).

The fact that a linear combination of two (or more) solutions of an equation is also a solution of the same equation, is called the *linear superposition principle.* It holds *only* for linear equations and does *not* hold for nonlinear ones. For example, if $u_1(\tau)$ and $u_2(\tau)$ are two solutions of (6.3), then

$$c_1u_1(\tau) + c_2u_2(\tau),$$

is *not*, in general (i.e. possibly except for some isolated, special values of c_1, c_2), a solution of (6.3), simply because

$$\sin(c_1u_1 + c_2u_2) \neq c_1\sin u_1 + c_2\sin u_2\,.$$

Now, returning to the linear model (6.4) and its solution (6.5), we note that c_1, c_2 are determined from the initial conditions. If $\theta(\tau = 0) = \theta_0$ and $\dot{\theta}(\tau = 0) = \Omega_0$, then (verify):

$$c_1 = \theta_0, \qquad c_2 = \Omega_0\,. \tag{6.6}$$

We will finish this section by presenting an alternative form of (6.5). Transform (6.5) as follows:

$$\begin{aligned} c_1\cos\tau + c_2\sin\tau &= \sqrt{c_1^2 + c_2^2}\left(\underbrace{\frac{c_1}{\sqrt{c_1^2+c_2^2}}}_{\cos\varphi}\cos\tau + \underbrace{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}_{\sin\varphi}\sin\tau\right) \\ &= \sqrt{c_1^2 + c_2^2}\,(\cos\tau\cdot\cos\varphi + \sin\tau\cdot\sin\varphi) \\ &= \sqrt{c_1^2 + c_2^2}\cdot\cos(\tau - \varphi)\,. \end{aligned}$$

Note that the definition of $\cos\varphi$ and $\sin\varphi$ made in the first line above makes sense because

$$\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right)^2 + \left(\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right)^2 = 1\,.$$

Now, since

$$\frac{\sin\varphi}{\cos\varphi} = \frac{\frac{c_2}{\sqrt{c_1^2+c_2^2}}}{\frac{c_1}{\sqrt{c_1^2+c_2^2}}} = \frac{c_2}{c_1} \overset{\text{see (6.6)}}{=} \frac{\theta_0}{\Omega_0}\,,$$

then we can write the solution (6.5), (6.6) equivalently as

$$\theta \;=\; A\cos(\tau-\varphi)$$
$$A = \sqrt{\theta_0^2+\Omega_0^2}\,, \qquad \varphi = \arctan\left(\frac{\theta_0}{\Omega_0}\right)\,. \tag{6.7}$$

Thus, (6.7) and (6.5), (6.6) are two equivalent forms of the solution of (6.4) satisfying the same initial conditions. The constant φ can be interpreted as a "time shift" (specifically, a delay), because $\cos(\tau-\varphi)$ is a replica of $\cos\tau$ shifted by φ units to the right.

6.5 Complex form of the solution; Euler formula

Above we have used the known properties of $\sin\tau$ and $\cos\tau$,

$$(\sin\tau)^{\cdot\cdot} = -\sin\tau, \qquad (\cos\tau)^{\cdot\cdot} = -\cos\tau\,,$$

to claim that these functions, and hence also their linear combination (6.5), solve (6.4). Below we will *derive* these facts from more basic principles.

As you know from the course on elementary differential equations, a solution of a differential equation with constant coefficients is sought in the form

$$u = e^{\lambda\tau} \tag{6.8}$$

for some $\lambda = \text{const}$. Substituting (6.8) into (6.4) and then cancelling by $e^{\lambda\tau} \neq 0$, we find

$$\lambda^2 + 1 = 0\,,$$

whence

$$\lambda = \pm\sqrt{-1} \equiv \pm i\,.$$

Therefore, $e^{\pm i\tau}$ are solutions of (6.4), and hence there must exist a connection between $e^{\pm i\tau}$ and $\sin\tau$, $\cos\tau$. This connection is established by using the Taylor expansion for $e^{i\tau}$. Before we establish it, however, we need some basic properties of the number i.

Since $(i)^2 = -1$, we have:

$$\begin{aligned}
(i)^3 &= (i)^2\cdot i = -i\,,\\
(i)^4 &= i^3\cdot i = -i\cdot i = -i^2 = -(-1) = 1\,,\\
(i)^5 &= i^4\cdot i = 1\cdot i = i\,,\\
(i)^6 &= i^5\cdot i = i\cdot i = -1\,, \qquad \text{etc}\,.
\end{aligned}$$

Now, the Maclaurin series for $e^{i\tau}$ is:

$$\begin{aligned}
e^{i\tau} &= 1 + (i\tau) + \frac{(i\tau)^2}{2!} + \frac{(i\tau)^3}{3!} + \frac{(i\tau)^4}{4!} + \frac{(i\tau)^5}{5!} + \dots\\
&= 1 + i\tau - \frac{\tau^2}{2!} - \frac{i\tau^3}{3!} + \frac{\tau^4}{4!} + \frac{i\tau^5}{5!} + \dots\\
&= \left(1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + \dots\right) + i\left(\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} + \dots\right)\\
&= \cos\tau + i\sin\tau\,,
\end{aligned}$$

where we have used the Maclaurin series for $\cos\tau$ and $\sin\tau$. This result is known as the

$$\textit{Euler formula:} \qquad \boxed{e^{i\tau} = \cos\tau + i\sin\tau}\,. \tag{6.9a}$$

At home you will prove that

$$e^{-i\tau} = \cos\tau - i\sin\tau\,. \tag{6.9b}$$

Since we have found earlier that $e^{i\tau}$ and $e^{-i\tau}$ are solutions of (6.4), then, by the linear superposition principle, so is their linear combination

$$\begin{aligned}\theta &= d_1 e^{i\tau} + d_2 e^{-i\tau} \qquad \text{(where } d_1,\ d_2 \text{ are some } \textit{complex-valued} \text{ constants)}\\ &= d_1(\cos\tau + i\sin\tau) + d_2(\cos\tau - i\sin\tau)\\ &= (d_1 + d_2)\cos\tau + (id_1 - id_2)\sin\tau\,,\end{aligned}$$

which is of the form (6.5). Thus, (6.5) has now been rigorously derived.

As we noted earlier, in (6.5), the constants c_1, c_2 are arbitrary real numbers (which need to be chosen to satisfy the initial conditions, but if no conditions are specified, then there is no restriction on c_1, c_2). On the contrary, in the expression above, the constants d_1 and d_2 must satisfy a certain condition to ensure that the solution θ is real-valued. To establish this condition, we need a few more facts about complex numbers.

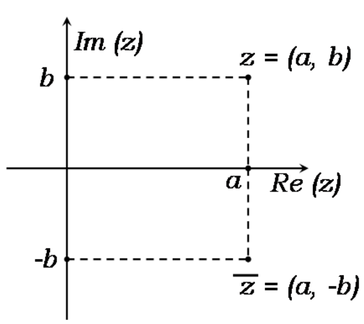
6.6 Some basic facts about complex numbers

Let

$$z = a + i \cdot b$$

be a complex number. One calls the *real* numbers a and b its real and imaginary parts, respectively:

$$\mathrm{Re}(z) = a, \qquad \mathrm{Im}(z) = b\,.$$



A complex number can be represented as a point in the plain, as shown on the left. The *complex conjugate* of $z = a + ib$ is $\overline{z} = a - ib$, i.e.

$$\mathrm{Re}(\overline{z}) = \mathrm{Re}(z), \qquad \mathrm{Im}(\overline{z}) = -\mathrm{Im}(z)\,.$$

(An alternative, and frequently used, notation for the complex conjugate of z is z^*, but we will use the notation $\overline{z}$.)

There are several simple corollaries of the definitions of z and $\overline{z}$. Namely,

$$z = a + ib \qquad \text{and} \qquad \overline{z} = a - ib$$

imply (verify all of the facts listed below):

$$a \equiv \mathrm{Re}(z) = \frac{(z + \overline{z})}{2}\,; \qquad b \equiv \mathrm{Im}(z) = \frac{(z - \overline{z})}{2i}\,; \tag{6.10a}$$

$$z \text{ is real} \quad \Leftrightarrow \quad \bar{z} = z\,; \tag{6.10b}$$

$$\bar{\bar{z}} = z\,. \tag{6.10c}$$

Another simple fact is that

$$\bar{i} = -i \tag{6.10d}$$

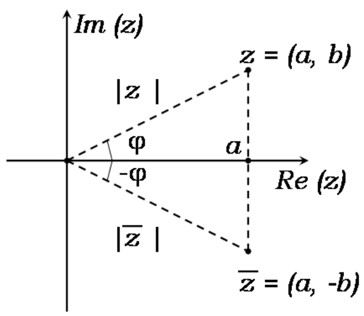
and also,

$$\frac{1}{i} = -i\,. \tag{6.10e}$$

At home you will also prove that if z and w are any two complex numbers, then

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}\,. \tag{6.11}$$

Now, as points in the plane have both cartesian and polar representations, so do complex numbers. Above we have used the Cartesian representation.



In the polar representation, one characterizes z by its distance from the origin:

$$|z| = \sqrt{a^2 + b^2}$$

and the angle ϕ such that

$$\tan\phi = \frac{b}{a}\,.$$

Quantities $|z|$ and ϕ are called the *modulus* and the *argument* of z.

The formula for the polar representation of z is obtained as follows:

$$\begin{aligned} z &= a + ib = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i \cdot \frac{b}{\sqrt{a^2+b^2}}\right) \\ &= |z| \cdot (\cos\phi + i\sin\phi) = |z|e^{i\phi}\,. \end{aligned}$$

Thus,

$$z = a + ib = |z|e^{i\phi}\,, \tag{6.12a}$$

$$|z| = \sqrt{a^2+b^2}\,, \qquad \phi = \arctan\left(\frac{b}{a}\right)\,.$$

(Which other formula, derived earlier in this lecture, does (6.12a) (and its derivation) remind you of?)

As follows from the figure above, if $z = |z|e^{i\phi}$, then

$$\bar{z} = |z|e^{-i\phi} \tag{6.12b}$$

(in particular, this implies $|\bar{z}| = |z|$). At home you will also demonstrate that

$$e^{\frac{i\pi}{2}} = i\,, \qquad e^{-\frac{i\pi}{2}} = -i\,, \tag{6.13}$$

which of course, is consistent with (6.10d,e).

With this background, we return to the ussue of finding a relation between constants d_1, d_2, introduced at the end of Sec. 6.5. We will also derive a complex form of the solution that is a counterpart of the real form (6.7).

6.7 An alternative complex form of the solution

Let us begin by rewriting the last equation of Sec. 6.5 as follows:

$$\theta = d_1 e^{i\tau} + d_2 e^{-i\tau} = d_1 e^{i\tau} + d_2 \overline{e^{i\tau}}\,, \tag{6.14a}$$

where we have used (6.12a) and (6.12b) with $|z| = 1$. Now, $\theta(\tau)$ is real-valued, so that $\overline{\theta} = \theta$ (see (6.10b)). Taking the complex conjugate of Eq. (6.14a) yields

$$\begin{aligned} \overline{\theta} &= \overline{d_1 e^{i\tau}} + \overline{d_2 \overline{e^{i\tau}}} \\ &\overset{(6.11)}{=} \overline{d_1} \cdot \overline{e^{i\tau}} + \overline{d_2} \cdot \overline{\overline{e^{i\tau}}} \\ &\overset{(6.12a,b),(6.10c)}{=} \overline{d_1} \cdot e^{-i\tau} + \overline{d_2} \cdot e^{i\tau}\,. \end{aligned} \tag{6.14b}$$

Comparing (6.14a) with (6.14b) and noticing that $\overline{\theta}$ must equal θ *for all* τ, we conclude that:

$$d_1 = \overline{d_2}, \qquad d_2 = \overline{d_1}\,.$$

Substituting this into (6.14a) and using (6.11), (6.12), we find (verify):

$$\theta = d_1 \cdot e^{i\tau} + \overline{d_1 \cdot e^{i\tau}} \equiv d_1 \cdot e^{i\tau} + \underbrace{c.c.}_{\text{complex conjugate}} \tag{6.14c}$$

Equation (6.14c) can be represented in yet another form. Recall that $z + \overline{z} = 2\mathrm{Re}(z)$ (see Eq. (6.10a)). Then (6.14c) becomes:

$$\theta = 2\mathrm{Re}(d_1 e^{i\tau})\,. \tag{6.14d}$$

Let us represent d_1 in polar form (6.12a):

$$d_1 = |d_1| e^{i\phi} \equiv \frac{1}{2} A e^{i\phi}.$$

(This defines the constants A and ϕ, and the "$\frac{1}{2}$" is used to cancel with the "2" in (6.14d).) Then (6.14d) becomes:

$$\theta \;=\; \mathrm{Re}(A e^{i(\tau+\phi)}) \;=\; A \cdot \mathrm{Re}(e^{i(\tau+\phi)}) \overset{(6.9a)}{=} A \cdot \cos(\tau + \phi), \tag{6.14e}$$

which, of course, is equivalent to (6.7) (with ϕ in (6.14e) being equal to $(-\varphi)$ in (6.7)).

Finally, we rewrite (6.14d) in a form that explicitly accounts for the initial conditions. Let

$$d_1 = \frac{1}{2}(a + ib)$$

(again, the "$\frac{1}{2}$" is here to cancel out the "2" in (6.14d)). Then

$$\begin{aligned} (a+ib)e^{i\tau} &= (a+ib)(\cos\tau + i\sin\tau) \\ &= a\cos\tau + i \cdot a \cdot \sin\tau + i \cdot b \cdot \cos\tau + i^2 \cdot b \cdot \sin\tau \\ &= (a\cos\tau - b\sin\tau) + i(a\sin\tau + b\cos\tau)\,. \end{aligned}$$

Substituting this into (6.14d) and using $\theta(\tau = 0) = \theta_0$ yields $a = \theta_0$. Taking the τ-derivative and setting $\dot{\theta}(\tau = 0) = \Omega_0$ yields $b = -\Omega_0$. Then (6.14d) becomes:

$$\theta = \mathrm{Re}((\theta_0 - i\Omega_0)e^{i\tau})\,. \tag{6.14f}$$

We will now demonstrate that the complex representations obtained are quite convenient for obtaining solutions of the harmonic oscillator model with friction and with an external force.

6.8 Harmonic oscillator with friction

In many cases, the friction can be assumed to be proportional to the velocity. (E.g., in a real pendulum, this will describe the viscous friction in the greased bearings.) The corresponding nondimensional model has the form

$$\ddot{\theta} + 2\gamma\,\dot{\theta} + \theta = 0\,, \tag{6.15}$$

where 2γ is the nondimensional friction coefficient and the "2" is put there for the convenience "down the road". Substituting formula (6.8) into (6.15), we obtain:

$$\lambda^2 + 2\gamma\lambda + 1 = 0,$$

whose solutions are

$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - 1}\,. \tag{6.16a}$$

One needs to distinguish two regimes, depending on the magnitude of the friction.

<u>$\gamma \geq 1$</u> Then both $\lambda_{1,2}$ are real and negative. In this case, the solution

$$\theta = c_1 e^{(-\gamma - \sqrt{\gamma^2-1}\,)\tau} + c_2 e^{(-\gamma + \sqrt{\gamma^2-1}\,)\tau} \tag{6.17a}$$

exhibits no oscillations. We will not consider this case further in the lecture, but you will be asked to explore one aspect of it in the homework.

<u>$\gamma < 1$</u> In this case, we can rewrite (6.16a) as:

$$\lambda_{1,2} = -\gamma \pm i\sqrt{1-\gamma^2} \equiv -\gamma \pm i\omega\,. \tag{6.16b}$$

Note that $\lambda_2 = \overline{\lambda_1}$ in this case, which implies (verify) $e^{\lambda_2\tau} = \overline{e^{\lambda_1\tau}}$. Then in complete analogy with the derivation of (6.14d) and (6.14e), one has

$$\begin{aligned} \theta &= 2\,\mathrm{Re}(d_1 \cdot e^{\lambda_1\tau}) \;=\; 2\,\mathrm{Re}(d_1 \cdot e^{(-\gamma+i\omega)\tau}) \\ &\equiv A \cdot e^{-\gamma\tau}\,\mathrm{Re}\, e^{i(\omega\tau+\phi)} \;=\; A \cdot e^{-\gamma\tau}\cos(\omega\tau+\phi)\,. \end{aligned} \tag{6.17b}$$

Thus, the amplitude of a harmonic oscillator with friction decays exponentially, as $e^{-\gamma\tau}$.

6.9 Forced oscillations; resonance

We will consider only the situation where the external force acting on the oscillator is a periodic function. The corresponding equation is

$$\ddot{\theta} + 2\gamma\dot{\theta} + \theta = F\cos(f\tau + \psi), \tag{6.18}$$

where F, f, and ψ are the amplitude, frequency, and phase of the force. Without loss of generality we can set $\psi = 0$. Indeed, if $\psi \neq 0$, then we can make it vanish by shifting the "time": $\cos(f\tau + \psi) = \cos(f(\tau + \frac{\psi}{f})) \equiv \cos(f \cdot \tau_{new})$.

As you know from the course on differential equations, the general solution of the *inhomogeneous* Eq. (6.18) is:

$$\theta_{\text{general}} = \theta_{\text{homogeneous}} + \theta_{\text{particular}}. \tag{6.19}$$

We have found the form of $\theta_{\text{homogeneous}}$ in Sec. 6.8 (see Eqs. (6.17)). Therefore now we will focus on finding a particular (i.e., some) solution of (6.18). We seek it in the form

$$\theta_{\text{particular}} \;=\; \frac{1}{2}(Pe^{if\tau} + c.c.) \;=\; \mathrm{Re}\,\left(P\,e^{if\tau}\right) \tag{6.20}$$

(see (6.14c) and (6.10a)), where the amplitude P may be complex. Substituting (6.20) into (6.18) (with $\psi = 0$), we find (verify):

$$\frac{1}{2}(P \cdot (-f^2)e^{if\tau} + c.c.) + 2\gamma \cdot \frac{1}{2}(P \cdot if \cdot e^{ift} + c.c.) + \frac{1}{2}(Pe^{ift} + c.c.) = \frac{1}{2}(Fe^{ift} + c.c.),$$

where in writing the r.h.s., we have used the formula

$$\cos x = \frac{1}{2}(e^{ix} + c.c.)\,,$$

which you will prove in the homework. Now, the previous equation (involving P) must hold *for all* τ, and therefore it suffices to collect all the terms multiplying e^{ift}:

$$P(-f^2 + 2\gamma \cdot if + 1) = F.$$

(Verify that the coefficients multiplying e^{-ift} will yield the same equation.) Thus from the above:

$$P = \frac{F}{(1-f^2) + 2i\gamma f}\,. \tag{6.21}$$

This derivation demonstrates the convenience of using the complex form of the solution to a linear differential equation.

We now investigate two particular cases of the solution given by Eqs. (6.19)–(6.21).

$\underline{\gamma \gg 1 \text{ (very large damping)}}$

Let also $f \neq 0$ (a nonconstant force). Then from (6.21),

$$P \approx \frac{F}{2i\gamma f} = \frac{F}{2\gamma f}e^{-\frac{i\pi}{2}},$$

where we have used Eqs. (6.10e) and (6.13). For large times, when the homogeneous solution decays (see Sec. 6.8), only the particular solution remains, which is (see (6.20)):

$$\theta \;\approx\; \theta_{\rm particular} \;\approx\; {\rm Re}\left(\frac{F}{2\gamma f}e^{if\tau - \frac{i\pi}{2}}\right) \;=\; \frac{F}{2\gamma f}\cos\left(f\tau - \frac{\pi}{2}\right) \;=\; \frac{F}{2\gamma f}\cos\left(f\big(\tau - \frac{\pi}{2f}\big)\right)\,.$$

This means that in this case, the pendulum follows the external force with the phase *delay* of $\pi/(2f)$.

$\underline{\gamma = 0,\ f = 1}$ (resonance)

In this case with no friction, the frequency of the external force exactly equals the natural frequency of the pendulum. Note that this case cannot be treated using (6.21) directly (why?). Also note that in the absence of friction ($\gamma = 0$), the homogeneous solution in (6.19) does not decay and hence may be as important as the particular solution. We will show below that it actually *is* important. The way to obtain the solution in this case is to consider either of the two limits:

(a) $f = 1,\ \gamma \to 0$;

(b) $f \to 1,\ \gamma = 0$.

Here we will consider limit (a), and at home you will reobtain the same result using limit (b).

Thus, we let $f = 1$ and $\gamma \to 0$. Then from (6.21), $P = F/(2i\gamma f)$. Combining this result with (6.20) and (6.19) and taking $\theta_{\text{homogeneous}}$ as the first line of (6.17b) with $d_1 = \frac{1}{2}(a+ib)$, we obtain:

$$\theta = \text{Re}\left((a+ib)e^{-\gamma\tau+i\omega\tau} + \frac{F}{2\gamma}\cdot\frac{1}{i}e^{i\tau}\right).$$

As before, a and b are found from the initial conditions. Namely, at $\tau = 0$:

$$\theta(\tau=0) = \theta_0 \qquad \Rightarrow \qquad \theta_0 = \text{Re}\left((a+ib) + \frac{F}{2\gamma}\cdot\frac{1}{i}\right) = a \tag{6.22a}$$

$$\dot{\theta}(\tau=0) = \Omega_0 \qquad \Rightarrow \qquad \Omega_0 = \text{Re}\left((a+ib)\cdot(-\gamma+i\omega) + \frac{F}{2\gamma}\cdot\frac{1}{i}\cdot i\right) = (-a\gamma - b\omega) + \frac{F}{2\gamma}. \tag{6.22b}$$

Solving (6.22a) and (6.22b) for a and b yields:

$$a = \theta_0, \qquad b = \frac{F}{2\gamma\omega} - \frac{\Omega_0}{\omega} - \frac{\theta_0\gamma}{\omega}. \tag{6.22c}$$

Now, as $\gamma \to 0$, $\omega = \sqrt{1-\gamma^2} \to 1$, and the most significant term in (6.22c) is seen to be $F/(2\gamma\omega) = O(1/\gamma)$. (Recall the O-notation from Lecture 3.) Substituting (6.22c) into the expression before (6.22a) and neglecting $O(\gamma)$ terms in comparison to the retained $O(1/\gamma)$ and $O(1)$ terms, we find:

$$\theta = \text{Re}\left((\theta_0 - i\Omega_0)e^{i\tau} + \frac{F}{2\gamma}\left[\frac{ie^{-\gamma\tau+i\omega\tau}}{\omega} + \frac{e^{i\tau}}{i}\right]\right). \tag{6.23}$$

In the first term on the r.h.s. of (6.23), which is $O(1)$, we also replaced $\omega = 1$ and $\gamma = 0$.

Let us consider the second term on the r.h.s. of (6.23) in detail. First, we observe that this is an indeterminate form 0/0, for when $\gamma \to 0$, the denominator clearly goes to zero, and the numerator does the same:

$$ie^{i\tau} + \frac{1}{i}e^{i\tau} = 0$$

(see (6.10e)). Second, we observe that ω can be set to 1. Indeed,

$$\omega = \sqrt{1-\gamma^2} = 1 - \frac{1}{2}\gamma^2 + O(\gamma^4) = 1 + O(\gamma^2),$$

whereas the indeterminate form we have is $O(\gamma)/O(\gamma)$, and hence $O(\gamma^2)$-terms can (and should) be omitted in the limit of $\gamma \to 0$. With this in mind, and also using (6.10e) and the Maclaurin expansion for $e^{-\gamma\tau}$, we rewrite the second term on the r.h.s. of (6.23) as:

$$\begin{aligned} \frac{F}{2\gamma}ie^{i\tau}(e^{-\gamma\tau} - 1) &= \frac{F}{2\gamma}ie^{i\tau}(\cancel{1} - \gamma\tau + O(\gamma^2) - \cancel{1}) \\ &= -\frac{F}{2}\,i\,\tau\,e^{i\tau} + O(\gamma). \end{aligned}$$

Thus, in the limit of $\gamma \to 0$, (6.23) takes on the form:

$$\begin{aligned} \theta &= \text{Re}\left((\theta_0 - i\Omega_0)e^{i\tau} - i\frac{F}{2}\tau e^{i\tau}\right) \\ &= (\theta_0\cos\tau + \Omega_0\sin\tau) + \frac{F}{2}\tau\sin\tau \end{aligned} \tag{6.24}$$

(verify). The last term describes oscillations with an amplitude that grows linearly in time. This linear growth of the oscillations when the frequency of the external force coincides with the natural frequency of the system, is called *resonance*.

6.10 Inverted pendulum

Let us return to Eq.(6.3), with no friction or external force. So far we have analyzed its solution near the equilibrium $\theta_E = 0$. This is a ***stable*** equilibrium, because *any solution that begins near it, keeps oscillating in its vicinity and does not "go away"*. We now consider the solution near the other equilibrium, $\theta_E = \pi$. We will show that this other equilibrium is ***unstable***, so that *almost any solution that starts near it, will eventually "go away"*.

We let $\theta = \pi + \theta^{(1)}$, where $\theta^{(1)}$ is a *small* deviation from the equilibrium. Then (6.3) yields:

$$(\pi + \theta^{(1)})^{\cdot\cdot} = -\sin(\pi + \theta^{(1)}),$$

or

$$\ddot{\theta^{(1)}} = \sin\theta^{(1)}. \tag{6.25}$$

We now use the assumption that $\theta^{(1)}$ is small and use the Maclaurin expansion for $\sin\theta^{(1)}$ (see Sec. 6.3). Keeping, as we did before, only the main-order term in that expansion, we reduce (6.25) to

$$\ddot{\theta^{(1)}} = \theta^{(1)}. \tag{6.26}$$

Following the approach of Sec. 6.5, we find that

$$\theta^{(1)} = c_1 e^{\tau} + c_2 e^{-\tau}. \tag{6.27}$$

Thus, unless $c_1 = 0$, which can happen for only one initial condition out of a continuum, the deviation $\theta^{(1)}$ will grow as e^{τ}. That is, the pendulum in the inverted position is unstable, as we know.

The take-home message of this subsection, which (the message) will be substantially referenced in Lecture 7, is as follows.

- Equation (6.4), or, more generally,

  $$\ddot{\theta} = (\textbf{negative}\ \text{constant}) \cdot \theta,$$

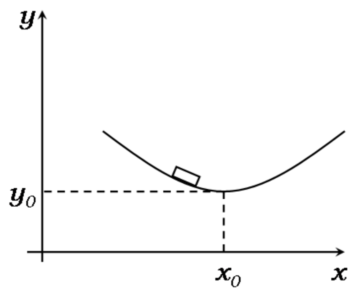
  describes a *stable oscillator*. This oscillator will behave according to its name: it will oscillate with a constant amplitude near the equilibrium point $\theta_E = 0$.

- On the contrary, Eq. (6.26), or, more generally,

  $$\ddot{\theta} = (\textbf{positive}\ \text{constant}) \cdot \theta,$$

  describes an *unstable pendulum*. We did not use the name "oscillator", because the corresponding physical object will *not oscillate* but, instead, will exponentially diverge from the equilibrium point $\theta_E = 0$.

6.11 Harmonic oscillator model for a block in a well



Finally, we will show that (6.4) also occurs (among many other situations) for a block placed near the bottom of a well. Mathematically, this is the same problem as considered in Lecture 4, so Eq. (4.10b) from that Lecture applies. We note that near the bottom of the well, its shape is approximated as:

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2} y_0'' + \dots \quad (6.28)$$

$$= y_0 + \frac{y_0''}{2} \cdot (x - x_0)^2 + \dots,$$

since $y_0' = 0$ at the bottom. Substituting (6.28) onto Eq. (4.10b) of Lecture 4 we obtain:

$$\frac{d^2x}{dt^2} = -\frac{(x - x_0)y_0''}{Q^2} \left(g + y_0'' \left(\frac{dx}{dt} \right)^2 \right), \quad (6.29)$$

where (see Lecture 4)

$$Q^2 = 1 + {y'}^2 = 1 + (y_0'' \cdot (x - x_0))^2 = 1 + O((x - x_0)^2) \, .$$

If we keep only terms of the order $O(x - x_0)$ (assuming that the deviation from the bottom is small), we should assume that $Q = 1$. By the same argument, we ignore the term $(\frac{dx}{dt})^2$ in (6.29). Then (6.29) reduces to:

$$\frac{d^2(x - x_0)}{dt^2} = -(g \cdot y_0'') \cdot (x - x_0). \quad (6.30)$$

Nondimensionalizing the time, as in Sec. 6.2, we confirm that (6.30) has the same form as (6.4).