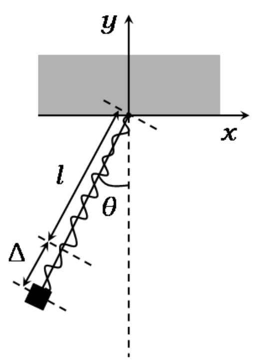
7 Pendulum. Part II: More complicated situations

In this Lecture, we will pursue two main goals. First, we will take a glimpse at a method of Classical Mechanics that allows one to obtain equations of motion in a *much* easier way than the Second Law of Newton does. We will apply this method to derive equations of motion in three cases where using the Second Law of Newton would be rather difficult. You will further practice this method in homework problems. However, we will *not* try to justify this powerful method – this is usually done in graduate courses of Classical Mechanics or Variational Calculus.

Second, we will continue to practice using linearization of equations near equilibrium states, where we will used the information obtained to analyze stability of these equilibria.

In one of the examples that we will consider, we will enounter a new form of a perturbation expansion, which will be conceptually different from the one we used in Lecture 3. Again, we will use this new form without any rigorous justification; such a justification is usually provided in a graduate course on the perturbation theory.

7.1 Motivation



Consider a pendulum shown on the left, where the mass is allowed to slide along the weightless rod that swings in a vertical plane. The mass is attached to the pivot point of the rod by a spring, so that the motion of the mass along the rod is restricted by what the spring "allows" the mass to do.

As you may suspect, it would be rather difficult to set up the Second Law of Newton in this case while keeping track of all the forces that act here. Fortunately, there is an easier method to obtain the equations of motion.

7.2 Euler-Lagrange equations

The ***Lagrangian*** of a system is defined as its kinetic energy *minus* the potential energy:

$$L = T - V. \tag{7.1}$$

Both the kinetic and the potential energy may depend on the ***coordinates*** that describe the system. We will denote those coordinates by $q_1, q_2, \ldots$. These are *not* necessarily the Cartesian coordinates, as we will see in the examples below.

The kinetic energy also depends on the velocities, and the potential energy usually does not. In this lecture we will use the notation $\dot{q} = \frac{dq}{dt}$, where t is the actual (dimensional) time. Thus, $T = T(q_1, q_2, \ldots; \dot{q}_1, \dot{q}_2, \ldots)$, $V = V(q_1, q_2, \ldots)$, so that

$$L = L(q_1, q_2, \ldots; \dot{q}_1, \dot{q}_2, \ldots).$$

Then the equations of motion for the coordinates, called ***Euler-Lagrange*** equations, are:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad i = 1, 2, \ldots \tag{7.2}$$

where $\partial L/\partial q_i$ is the *partial* derivative of L with respect to q_i. Recall that when taking the partial derivative with respect to q_i, all the other variables (*including* $\dot{q}_i$!) must be treated as constants.

Verify (*yes, do it*) that the dimensional units of both terms in (7.2) are the same. (Otherwise, the equation would not make sense.)

We will now apply Eq. (7.2) to three different modifications of the simple pendulum problem considered in the previous lecture.

7.3 Pendulum on a spring

Let us return to the problem described in Sec. 7.1. Let l be the natural length of the spring and Δ be the amount by which the mass compresses or stretches the spring. Thus, $(l + \Delta)$ is the total length of the spring. Also, let θ be the angle by which the rod deviates from the vertical. Finally, let x and y be the Cartesian coordinates of the mass, where the Cartesian origin is at the pivot point (see the figure in Sec 7.1).

The kinetic energy of the mass equals:

$$T = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} \,. \tag{7.3a}$$

The potential energy is the sum of the gravitational potential energy and the spring potential energy:

$$V = mgy + \frac{k\Delta^2}{2} \,. \tag{7.3b}$$

Thus, using (7.1), one has that

$$L = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} - mgy - \frac{k\Delta^2}{2} \,. \tag{7.3c}$$

We will now express x and y via l and Δ. From the aforementioned figure, we obtain:

$$x = (l + \Delta)\sin\theta, \qquad y = -(l + \Delta)\cos\theta, \tag{7.4a}$$

$$\dot{x} = (l + \Delta)\dot{\theta}\cos\theta + \dot{\Delta}\sin\theta, \qquad \dot{y} = (l + \Delta)\dot{\theta}\sin\theta - \dot{\Delta}\cos\theta. \tag{7.4b}$$

Then:

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 &= (l+\Delta)^2\dot{\theta}^2\cos^2\theta + 2(l+\Delta)\dot{\theta}\cos\theta\dot{\Delta}\sin\theta + \dot{\Delta}^2\sin^2\theta + \\ &\quad (l+\Delta)^2\dot{\theta}^2\sin^2\theta - 2(l+\Delta)\dot{\theta}\sin\theta\dot{\Delta}\cos\theta + \dot{\Delta}^2\cos^2\theta \\ &= (l+\Delta)^2\dot{\theta}^2 + \dot{\Delta}^2. \end{aligned}$$

Substituting the last expression and y from (7.4a) into (7.3c), we obtain:

$$L = \frac{m}{2}\left((l+\Delta)^2\dot{\theta}^2 + \dot{\Delta}^2\right) + mg(l+\Delta)\cos\theta - \frac{k\Delta^2}{2}. \tag{7.5}$$

The "coordinates" q_1, q_2 in this Lagrangian are:

$$q_1 = \theta, \qquad q_2 = \Delta.$$

Substituting (7.5) into (7.2) yields:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \qquad \Rightarrow$$

$$-mg(l+\Delta)\sin\theta - \frac{d}{dt}\left(\frac{m}{2}(l+\Delta)^2\cdot 2\dot\theta\right) = 0 \qquad \Rightarrow$$

$$-g(l+\Delta)\sin\theta - [2(l+\Delta)\dot\Delta\dot\theta + (l+\Delta)^2\cdot\ddot\theta] = 0 \qquad \Rightarrow$$

$$g\sin\theta + 2\dot\Delta\dot\theta + (l+\Delta)\ddot\theta = 0. \tag{7.6a}$$

(Recall that when taking $\partial/\partial\theta$, the variables $\dot\theta, \Delta, \dot\Delta$ are treated as constants, and similarly for $\partial/\partial\dot\theta$, $\partial/\partial\Delta$, $\partial/\partial\dot\Delta$.)

The other Euler-Lagrange equation is:

$$\frac{\partial L}{\partial\Delta} - \frac{d}{dt}\frac{\partial L}{\partial\dot\Delta} = 0 \qquad \Rightarrow$$

$$\left[\frac{m}{2}\cdot 2(l+\Delta)\dot\theta^2 + mg\cos\theta - k\Delta\right] - \frac{d}{dt}\left[\frac{m}{2}\cdot 2\dot\Delta\right] = 0 \qquad \Rightarrow$$

$$(l+\Delta)\,\dot\theta^2 + g\cos\theta - \frac{k}{m}\Delta - \ddot\Delta = 0. \tag{7.6b}$$

The coupled equations (7.6a) and (7.6b) describe the motion of the pendulum on a spring.

Before we proceed with their analysis, we need to check some limiting case(s) where we know the result. Such a case is that of a very tight spring (i.e., $k \to \infty$), where we expect these two equations to reduce to the equation of a simple pendulum for θ, and to $\Delta = 0$. The latter equation indeed follows from (7.6b) rewritten as:

$$\Delta = \frac{m}{k}\left[(l+\Delta)\dot\theta^2 + g\cos\theta - \ddot\Delta\right] \to 0 \quad \text{as} \quad k \to \infty.$$

Then, Eq. (7.6a) with $\Delta = 0$ reduces to Eq. (6.2) of Lecture 6 (verify).

The coupled nonlinear second-order equations (7.6) probably cannot be solved analytically. However, we can analyze them near the equilibrium states of the system by means of linearization. Recall that at an equilibrium, all time derivatives vanish identically:

$$\dot\Delta = \dot\theta = \ddot\Delta = \ddot\theta = 0.$$

Using this condition in Eqs. (7.6) yields:

$$\sin\theta = 0 \qquad \text{and} \qquad g\cos\theta = \frac{k}{m}\Delta, \tag{7.7a}$$

which implies two different equilibria:

$$\left(\theta_E = 0,\ \Delta_E = \frac{mg}{k}\right) \qquad \text{and} \qquad \left(\theta_E = \pi,\ \Delta_E = -\frac{mg}{k}\right). \tag{7.7b}$$

What is the physical meaning of these formulae (i.e., can you see how they follow from elementary considerations)?

We now linearize Eqs. (7.6) near each equilibrium, to which end we take

$$\begin{aligned} \theta &= \theta_E + \varphi, \qquad & \varphi \ll 1; \\ \Delta &= \Delta_E + \delta, \qquad & \delta \ll \Delta_E. \end{aligned}$$

We substitute (7.8) into (7.6) and keep only terms $O(\varphi)$ and $O(\delta)$; all higher-order terms are neglected. For the first equilibrium in (7.7b), one has:

$$\sin\theta = \sin\varphi \approx \varphi,$$

$$\cos\theta = \cos\varphi = 1 + O(\varphi^2),$$
$$\dot{\Delta} = O(\delta), \qquad \dot{\theta} = O(\varphi).$$

Then:

$$\underline{(7.6\text{a})} \quad \Rightarrow \quad g\varphi + O(\varphi\cdot\delta) + (l + \Delta_E + \delta)\ddot{\varphi} = 0$$
$$\Rightarrow \quad \ddot{\varphi} = -\left(\frac{g}{l+\Delta_E}\right)\varphi, \qquad (7.9a)$$
$$\underline{(7.6\text{b})} \quad \Rightarrow \quad O(\delta^2) + g - \frac{k}{m}(\Delta_E + \delta) - \ddot{\delta} = 0$$
$$\Rightarrow \quad \ddot{\delta} = -\frac{k}{m}\delta. \qquad (7.9b)$$

In obtaining the final equations in both (7.9a) and (7.9b), we have omitted second-order small terms. Moreover, in deriving the second line of (7.9b) from the previous line, we used the expression for Δ_E from (7.7b).

Let us repeat the last sentence, as the corresponding step is very important and will occur every time we derive linearized equations. So, *when obtaining first-order equations for small deviations about an equilibrium, one always uses the zeroth-order equation(s) satisfied by the system variables in this equilibrium.* For example, in deriving the linearizations equations (7.9), we used the equilibrium equations stated in the first parentheses in (7.7b).

Let us now continue with the analysis of Eqs. (7.9). Equation (7.9a) is precisely the equation of small oscillations of a pendulum of length $(l + \Delta_E)$; see Eqs. (6.2) and (6.4) in Lecture 6. Equation (7.9b) is precisely the equation of (small) oscillations of mass m on a spring (as derived in a physics course). From these statements we make two conclusions:

(i) *Small* oscillations of the pendulum on a spring *decouple* into two *independent modes* of motion: the oscillations of a rigid pendulum and the oscillations of a mass on a non-swinging spring.

It should be noted that such a decoupling of complex oscillations into basic modes does *not* occur always; it is even true that *in general*, it would *not* occur.

(ii) Since both Eqs. (7.9a) and (7.9b) have the form of Eq. (6.4) of Lecture 6, which describes stable oscillations, then the small oscillations of the pendulum on a spring near the first equilibrium in (7.7b) (i.e., the "down" position), are stable.

Now consider the small deviation of the pendulum from the second equilibrium (7.7b). Then

$$\sin\theta = \sin(\pi + \varphi) = -\sin\varphi \approx -\varphi$$
$$\cos\theta = \cos(\pi + \varphi) = -\cos\varphi \approx -1$$
$$\dot{\Delta} = O(\delta)\ , \qquad \dot{\theta} = O(\varphi).$$

Similarly to (7.9a) and (7.9b), one obtains (verify):

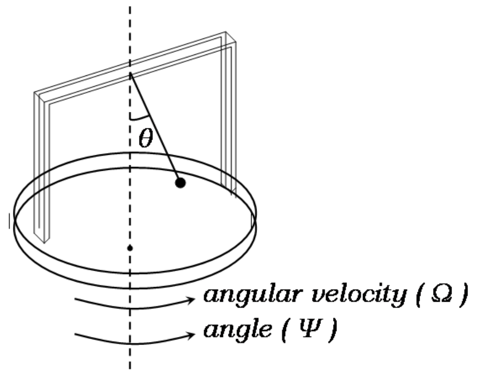
$$-g\varphi + (l + \Delta_E)\ddot{\varphi} = 0 \qquad (7.10a)$$
$$-\frac{k}{m}\delta - \ddot{\delta} = 0. \qquad (7.10b)$$

In obtaining (7.10b) we have used the definition of Δ_E for the second equilibrium in (7.7b).

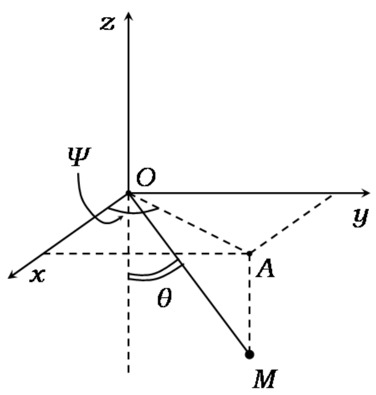
Conclusion (i) above holds in this case also: the rotational motion of the pendulum and the vibrations of the mass on the spring are decoupled. But while the latter motion remains stable (note that (7.10b) is equivalent to (7.9b)), the rotational motion is now unstable. Indeed, (7.10a) is an analog of Eq. (6.26) of Lecture 6 – it describes the unstable motion of an inverted pendulum. This is precisely what one would intuitively expect in this case.

7.4 Pendulum on a rotating platform

A problem that we will solve in this section is as follows.



A gate is attached to a platform that rotates uniformly with angular velocity $\Omega = \dot{\psi}$ about a vertical axis. A pendulum is attached to the gate so that it can move in a plane perpendicular to that of the gate, and its pivot point is on the axis of rotation of the platform. The mass of the bob is m and the length of the rod is l. Find the equation of motion of the pendulum and investigate the stability of its solution near the equilibrium state(s).



The location of the bob is now characterized by *three* Cartesian coordinates x, y, z:

$$z = -|OM| \cdot \cos\theta$$

$$x = |OA| \cdot \cos\psi$$

$$y = |OA| \cdot \sin\psi\,.$$

Note that $|OM|$, θ, and ψ are just the *spherical* coordinates of point M. Also,

$$|OA| = |OM| \cdot \sin\theta.$$

Finally, using the given information that $|OM| = l$ and $\psi = \Omega t$, one has:

$$\begin{aligned} x &= l \cdot \sin\theta\cos(\Omega t) \\ y &= l \cdot \sin\theta\sin(\Omega t) \\ z &= -l\cos\theta\,. \end{aligned} \tag{7.10}$$

The kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,, \tag{7.12a}$$

and the potential energy is simply

$$V = mgz\,. \tag{7.12b}$$

To compute T, we first compute the time derivatives of the Cartesian coordinates (7.10):

$$\begin{aligned}\dot{x} &= l[\dot{\theta}\cos\theta\cos(\Omega t) - \sin\theta\cdot\Omega\sin(\Omega t)]\\ \dot{y} &= l[\dot{\theta}\cos\theta\sin(\Omega t) + \sin\theta\cdot\Omega\cos(\Omega t)]\\ \dot{z} &= l\cdot\dot{\theta}\sin\theta.\end{aligned} \tag{7.12}$$

Substituting formulae (7.12) into (7.12a), one finds (verify):

$$T = \frac{ml^2}{2}(\Omega^2\sin^2\theta + \dot{\theta}^2).$$

Combining this expression with (7.12b), the last line of (7.10), and (7.1) one has:

$$L = \frac{ml^2}{2}(\Omega^2\sin^2\theta + \dot{\theta}^2) + mgl\cos\theta. \tag{7.13}$$

This Lagrangian has only *one* coordinate: θ. Then the Euler-Lagrange equation is:

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0 \qquad \Rightarrow$$

$$\frac{ml^2}{2}\cdot\Omega^2\cdot 2\sin\theta\cos\theta - mgl\sin\theta - \frac{d}{dt}\left(\frac{ml^2}{2}\cdot 2\dot{\theta}\right) = 0 \qquad \Rightarrow$$

$$\ddot{\theta} = \Omega^2\sin\theta\cos\theta - \frac{g}{l}\sin\theta.$$

Introducing the notations

$$\omega_0^2 = \frac{g}{l}, \qquad R = \frac{\Omega}{\omega_0}, \tag{7.14}$$

the previous equation can be rewritten as

$$\ddot{\theta} = \omega_0^2\sin\theta(R^2\cos\theta - 1). \tag{7.15}$$

This is the equation for the pendulum on a rotating platform. We now find its equilibria and then linearize about them.

The equilibria of (7.15) occur where $\ddot{\theta} = 0$, and hence the r.h.s. of (7.15) vanishes. This occurs in three distinct cases:

a) $\sin\theta = 0 \quad \Rightarrow \quad \theta_E = 0;$
b) $\sin\theta = 0 \quad \Rightarrow \quad \theta_E = \pi;$
c) $R^2\cos\theta - 1 = 0 \quad \Rightarrow \quad \theta_E = \pm\arccos\left(\frac{1}{R^2}\right).$

Note that the two states in case c) exist *only* for $R > 1$. From the symmetry considerations, these states are physically equivalent, since one can be obtained from the other by rotating the platform by 180°. Therefore, below we will consider only one of these states, say,

$$\theta_E = +\arccos\left(\frac{1}{R^2}\right).$$

As in Eq. (7.8), we take

$$\theta = \theta_E + \varphi\,, \qquad \varphi \ll 1.$$

In case a), we have:

$$\sin\theta = \sin\varphi \approx \varphi, \qquad \cos\theta = \cos\varphi \approx 1\,.$$

Then the linearized version of (7.15) becomes

$$\ddot{\varphi} = \omega_0^2 \cdot \varphi(R^2 - 1). \tag{7.17a}$$

Thus the "down" position of the pendulum is stable when $R^2 \leq 1$ and unstable otherwise.

In case b), we have:

$$\sin\theta = \sin(\pi + \varphi) \approx -\varphi$$
$$\cos\theta = \cos(\pi + \varphi) \approx -1\,,$$

whence

$$\ddot{\varphi} = -\omega_0^2 \cdot \varphi \cdot (-R^2 - 1) \qquad \Rightarrow \qquad \ddot{\varphi} = \omega_0^2 \cdot \varphi \cdot (R^2 + 1). \tag{7.17b}$$

Thus, the "up" position is always unstable.

In case c), we use the first two terms of the Taylor series for $\sin(\theta_E + \varphi)$ and $\cos(\theta_E + \varphi)$:

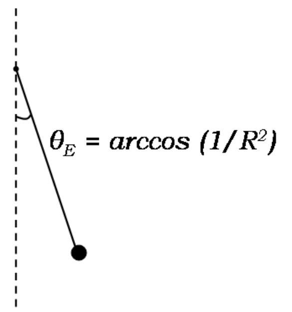
$$\sin\theta = \sin(\theta_E + \varphi) \approx \sin\theta_E + \varphi\cos\theta_E,$$
$$\cos\theta = \cos(\theta_E + \varphi) \approx \cos\theta_E - \varphi\sin\theta_E,$$

whence:

$$\ddot{\varphi} = \omega_0^2(\sin\theta_E + \varphi\cos\theta_E)\big(R^2[\underline{\cos\theta_E} - \varphi\sin\theta_E] - \underline{1}\big)\,.$$

Now, recall the step that we emphasized after deriving Eqs. (7.9). Accordingly, we use the equation satisfied by the equilibrium angle θ_E in case c): $(R^2\cos\theta_E - 1) = 0$. Then we see that the underlined terms in the equation above cancel out. Finally, neglecting terms $O(\varphi^2)$, as before, we obtain (verify):

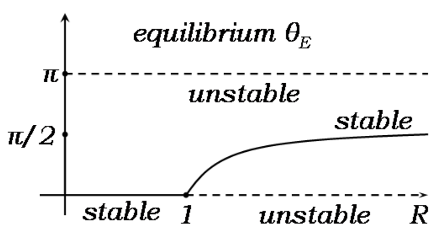
$$\ddot{\varphi} = -\omega_0^2\, R^2 \sin^2\theta_E \cdot \varphi\,. \tag{7.17c}$$



Since the coefficient in front of φ on the r.h.s. is always negative, we conclude that this "tilted" equilibrium is stable whenever it exists, i.e. for $R > 1$.
In other words, this "tilted" equilibrium exists, and is stable, when $\Omega > \omega_0$ (see (7.14)), i.e. the angular speed of the platform exceeds the natural frequency of a simple pendulum.

Using the expression $\sin^2\theta_E = 1 - \cos^2\theta_E = \left(1 - \frac{1}{R^4}\right)$ and $R^2 = (\Omega^2/\omega_0^2)$, Eq. (7.17c) can be rewritten as (verify):

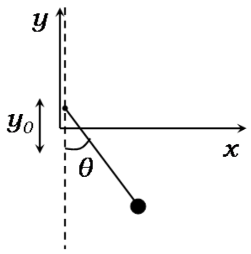
$$\ddot{\varphi} = -\left(\Omega^2 - \frac{\omega_0^4}{\Omega^2}\right)\varphi. \tag{7.17c$'$}$$



To conclude the consideration of this physical system, we summarize its equilibrium states and their stability in a so-called bifurcation diagram, shown on the left. The solid (dashed) lines depict the stable (unstable) equilibria as functions of the *bifurcation parameter* R.

7.5 Pendulum with the pivot point rapidly oscillating in the vertical direction

In this example, we will find that a pendulum described in the title of this section *can* be stable in the "up" position! This physical system was first analysed in 1951 (yes, so recently!) by a great Russian physicist Pyotr Kapitza (Kapitsa). It is a particular case of a more general situation where a system whose natural time scale is "of order one" (i.e. is neither too fast or too slow) is affected by a fast periodic force. Some links to applications of such systems in physics (and not just in mechanics) are pointed out on the course web page.



Denote the coordinate of the pivot point

$$y_0(t) = -a\cos\Omega t$$

(the "minus" sign is just for the convenience later on). Thus, a and Ω are the amplitude and frequency of the pivot's vibrations. Clearly,

$$x = l\sin\theta\,, \qquad (7.18a)$$

$$y = -l\cos\theta + y_0(t) = -(l\cos\theta + a\cos\Omega t). \qquad (7.18b)$$

The kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2).$$

From (7.18), we find:

$$\dot{x} = l\,\dot{\theta}\cos\theta \qquad (7.19a)$$

$$\dot{y} = l\,\dot{\theta}\sin\theta + a\,\Omega\sin\Omega t\,. \qquad (7.19b)$$

The potential energy is

$$V = mgy.$$

Combining these equations and Eq. (7.1), we obtain the Lagrangian (verify):

$$L = \frac{m}{2}(\,l^2\dot{\theta}^2 + 2\,a\,l\,\dot{\theta}\,\Omega\sin\theta\,\sin\Omega t + \Omega^2\cdot a^2\sin^2\Omega t\,) + mgl\cos\theta + mga\cos\Omega t\,. \qquad (7.19)$$

The only coordinate in this Lagrangian is the angle θ. The Euler-Lagrange equation is:

$$\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \qquad \Rightarrow$$

$$\frac{m}{2}\cdot 2\,a\,l\,\dot{\theta}\,\Omega\cos\theta\,\sin\Omega t - mgl\sin\theta - \frac{d}{dt}\left(\frac{ml^2}{2}\cdot 2\dot{\theta} + \frac{m}{2}\cdot 2\,a\,l\,\Omega\sin\theta\,\sin\Omega t\right) = 0\,.$$

(Note that the two terms in the Lagrangian that depend only on time but not on θ and $\dot{\theta}$, do *not* contribute to the equation of motion.)

Simplifying the above equation, we have:

$$\underline{a\,l\,\dot{\theta}\,\Omega\cos\theta\,\sin\Omega t} - gl\sin\theta - (\,l^2\ddot{\theta} + \underline{a\,l\,\Omega\,\dot{\theta}\cos\theta\,\sin\Omega t} + a\,l\,\Omega^2\sin\theta\,\cos\Omega t\,) = 0 \qquad \Rightarrow$$

$$-g\sin\theta = l\ddot{\theta} + a\,\Omega^2\sin\theta\,\cos\Omega t\,, \qquad \Rightarrow$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta\cdot\left(1 + \frac{a}{l}\cdot\frac{l}{g}\Omega^2\cos\Omega t\right)\,.$$

Using notations (7.14) and also introducing a new notation

$$\epsilon = \frac{a}{l}\,, \tag{7.20}$$

the last equation is rewritten as:

$$\ddot{\theta} = -\omega_0^2 \sin\theta \cdot (1 + \epsilon\, R^2 \cos\Omega t)\,. \tag{7.22a}$$

Since the analysis of this problem will be longer than the analysis of the two previous problems, we introduce one more notation: as in Lecture 6, we define the nondimensional "time"

$$\tau = \omega_0\, t$$

(so that $\Omega t = \frac{\Omega}{\omega_0}\omega_0 t = R\tau$). With this change of variables, (7.22a) becomes (verify):

$$\frac{d^2}{d\tau^2}\theta = -\sin\theta(1 + \epsilon\, R^2 \cos R\tau)\,. \tag{7.22b}$$

In what follows we will analyze Eq. (7.22b). Let us stress that the method we will use is *mathematically incorrect*; it is a physicist's method based mostly on intuition (originally – on that of the great physicist Kapitza). Both establishing a mathematically correct method for this problem and explaining *why* the method we will use is incorrect, belong to the material of a graduate course on perturbation theory. (E.g., solving (7.22b) in a mathematically correct way is assigned as Problem 6 for Sec. 6.4 in M.H. Holmes, Introduction to perturbation methods, Springer, New York, 1998.)

The following analysis is based on two assumptions:

(i) The amplitude of the pivot's vibrations is small compared to the length of the pendulum:

$$\frac{a}{l} \equiv \epsilon \ll 1\,; \tag{7.23a}$$

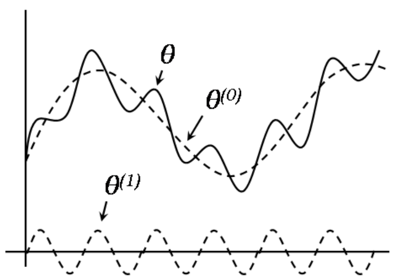
(ii) The frequency of the pivot's vibrations is large compared to the natural frequency of the pendulum:

$$\frac{\Omega}{\omega_0} \equiv R \gg 1\,. \tag{7.23b}$$

We do not specify any relation between ϵ and R at this moment; it will become apparent later.

Now, since the amplitude of the pivot's vibrations is small, Kapitza sought the solution of (7.22b) in the form:

$$\theta = \underbrace{\theta^{(0)}}_{\substack{\text{solution varying on the} \\ \text{time scale of the natural period} \\ \text{of the pendulum, } T = 2/\pi\omega_0}} + \underbrace{\epsilon\,\theta^{(1)}}_{\substack{\text{fast and small oscillations} \\ \text{superimposed on the} \\ \text{slower solution } \theta^{(0)}}} \,. \tag{7.23}$$



The fast oscillations can be assumed to be small because the vibrations of the pivot are small.

Note that *unlike in Lecture 3*, here the slower solution $\theta^{(0)}$ is *not* just the solution of a pendulum with a fixed pivot. All that we assume about $\theta^{(0)}$ is that it does not have any fast components.

Preparing to substitute (7.23) into (7.22b), we write:

$$\sin\theta = \sin(\theta^{(0)} + \epsilon\,\theta^{(1)}) \approx \sin\theta^{(0)} + \epsilon\,\theta^{(1)}\cos\theta^{(0)}\,.$$

This approximation comes from the same Taylor series as used in Case c) in Sec. 7.4, where now the role of $\varphi \ll 1$ is played by $\epsilon\theta^{(1)} \ll 1$ (see (7.23a) and (7.23)). Substituting the last equation and Eq. (7.23) into (7.22b), we have:

$$\frac{d^2\theta^{(0)}}{d\tau^2} + \epsilon\,\frac{d^2\theta^{(1)}}{d\tau^2} = -(\sin\theta^{(0)} + \epsilon\,\theta^{(1)}\cos\theta^{(0)})(1 + \epsilon R^2\cos R\tau)\,. \qquad (7.24)$$

In (7.24) let us separate the fast terms from the slower ones. Note that this, again, is *different from the perturbation theory of Lecture 3*, where we separated terms based on their size and *not* on their time scale.

Fast terms varying as $\cos R\tau$:

$$\frac{d^2\theta^{(1)}}{d\tau^2} = -\sin\theta^{(0)}\cdot R^2\cos R\tau - \theta^{(1)}\cos\theta^{(0)}\,, \qquad (7.26a)$$

Slower terms:

$$\frac{d^2\theta^{(0)}}{d\tau^2} = -\sin\theta^{(0)} - \underline{\epsilon\theta^{(1)}\cos\theta^{(0)}\cdot\epsilon R^2\cos R\tau}\,. \qquad (7.26b)$$

Why is the underlined term in (7.26b), which is the product of two fast terms, $\theta^{(1)}$ and $\cos(R\tau)$, included into the equation for the slower terms?? The answer will come soon.

Consider first the "fast equation" (7.26a). This is a linear differential equation with the forcing term. Moreover, on the time scale of the fast oscillations, the slower function $\theta^{(0)}$ changes very little (usually, much less than schematically shown in the figure above). Therefore, in the "fast equation", the slower variable $\theta^{(0)}$ can be considered as a constant.

Now, instead of solving (7.26a) exactly, we employ the following hand-waving argument[9]: Let us *disregard* the second term on the r.h.s., because it is $O(1)$, while the first term is $O(R^2) \gg 1$ (see (7.23b)). Then we simply have:

$$\frac{d^2\theta^{(1)}}{d\tau^2} = -\sin\theta^{(0)}\cdot R^2\cdot\cos R\tau\,,$$

whose solution is

$$\theta^{(1)} = \sin\theta^{(0)}\cdot\cos R\tau\,. \qquad (7.26)$$

When verifying (7.26), recall that in the "fast equation" (7.26a), $\theta^{(0)}$ is to be treated as a constant on the "fast" time scale.

Substituting (7.26) into the "slower equation" (7.26b), we find:

$$\frac{d^2\theta^{(0)}}{d\tau^2} = -\sin\theta^{(0)} - \epsilon^2R^2\cos\theta^{(0)}\sin\theta^{(0)}\cos^2 R\tau\,. \qquad (7.27)$$

We now recall the trigonometric identity

$$\cos^2(R\tau) = \frac{1+\cos(2R\tau)}{2}$$

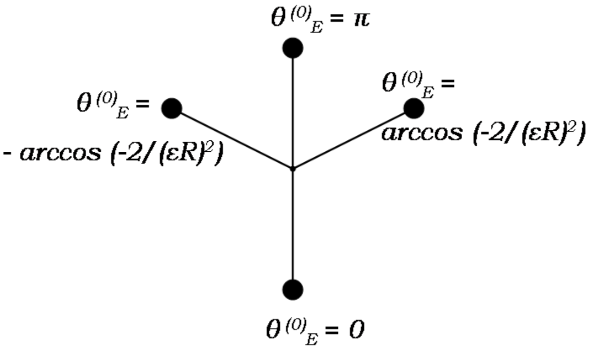
[9] Again, recall that we are handling this problem using physical intuition rather than mathematical rigor.

and discard the "fast" term $\cos(2R\tau)$ since *by design*, Eq. (7.27) is supposed *not* to have fast terms. Then (7.27) becomes:

$$\frac{d^2\theta^{(0)}}{d\tau^2} = -\sin\theta^{(0)}\left(1 + \frac{(\epsilon R)^2}{2}\cos\theta^{(0)}\right). \tag{7.28}$$

Note that (7.28) contains only the slower variable $\theta^{(0)}$. Also note that this equation is *different* from the equation of a pendulum with a fixed pivot point, Eq. (6.3) of Lecture 6, as we have already announced after Eq. (7.23). The difference, i.e. the second term on the r.h.s. of (7.28), comes from the coupling of the fast part of the solution, $\theta^{(1)}$, with the external force $\cos(R\tau)$; see Eq. (7.26b).

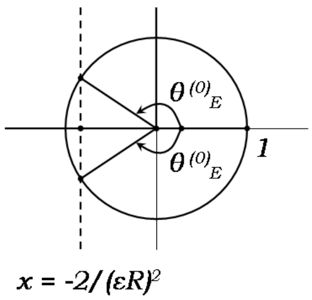
We now look for the equilibrium states of this equation and then will consider their stability. As earlier for Eq. (7.15), for Eq. (7.28) we also have three different cases:



a) $\sin\theta^{(0)} = 0 \quad \Rightarrow \quad \theta_E^{(0)} = 0;$

b) $\sin\theta^{(0)} = 0 \quad \Rightarrow \quad \theta_E^{(0)} = \pi;$

c) $1 + \frac{(\epsilon R)^2}{2}\cos\theta^{(0)} = 0 \quad \Rightarrow \quad \theta_E^{(0)} = \pm\arccos\left(-\frac{2}{(\epsilon R)^2}\right).$



As explained in the corresponding place in Sec. 7.4, we can consider only one of the two equilibria in case c), say,

$$\theta_E^{(0)} = \arccos\left(-\frac{2}{(\epsilon R)^2}\right).$$

It corresponds to the upper point of intersection between the dashed line and the circle in the figure on the left.

A study of the stability of these equilibria proceeds as in Sec. 7.4. Below I will present only the final equations (followed by their interpretations); you are responsible for verification of the details.

In case a),

$$\frac{d^2\varphi}{d\tau^2} = -\varphi\cdot\left(1 + \frac{(\epsilon R)^2}{2}\right). \tag{7.30a}$$

Thus, the "down" position of this pendulum is stable for all amplitudes and frequencies of the pivot's vibrations (at least as long as assumptions (7.23) hold).

In case b):

$$\frac{d^2\varphi}{d\tau^2} = \varphi\left(1 - \frac{(\epsilon R)^2}{2}\right) . \qquad (7.30b)$$

Thus, the "up" position of this pendulum is unstable when $(\epsilon R)^2 < 2$ and stable when $(\epsilon R)^2 > 2$. That is, sufficiently, fast vibrations, with

$$\frac{\Omega}{\omega_0} > \sqrt{2} \cdot \frac{l}{a} \, ,$$

would stabilize the pendulum in its inverted position!

In case c):

$$\frac{d^2\varphi}{d\tau^2} = -(\sin\theta_E^{(0)} + \varphi\cos\theta_E^{(0)}) \left(\underbrace{1 + \frac{(\epsilon R)^2}{2}\cos\theta_E^{(0)}}_{=0} - \frac{(\epsilon R)^2}{2}\cdot\varphi\,\sin\theta_E^{(0)} \right) = \frac{(\epsilon R)^2}{2}\cdot\sin^2\theta_E^{(0)}\cdot\varphi + O(\varphi^2) \, . \qquad (7.30c)$$

(The terms with the underbracket cancel out by the argument emphasized after Eq. (7.9b).) Thus, the "tilted" equilibria of this pendulum are always (i.e. for any vibration frequency) unstable.

Finally, we summarize the above conclusions in a bifurcation diagram for this physical system: