8 Pendulum. Part III: Higher-order corrections and perturbation theory

In this lecture we will pursue two main goals. First, we will use the Taylor expansion to derive models beyond linear (e.g., harmonic oscillator) ones. Second, we will treat the nonlinear terms as small perturbations and attempt to apply the regular perturbation theory analogous to that used in Lecture 3. However, we will discover that such a perturbation theory is inadequate in this case. We will investigate a reason for that and propose a direction in which an alternative perturbation theory should be developed. However, a proper development of such a theory is a subject of a graduate course and will not be considered here.

8.1 First-order correction to the harmonic oscillator model for a pendulum, and an attempt at using a regular perturbation theory for it

Recall the equation of a simple pendulum (Eq. (6.2) in Lecture 6):

$$\ddot{\theta} = -\omega_0^2 \sin\theta \,, \tag{8.1}$$

where the overdot denotes the time derivative and $\omega_0^2 = g/l$. As before, consider the angle θ to be small:

$$\theta = 0 + \varphi, \qquad \varphi \ll 1$$

and substitute this into (8.1), but unlike what we did in Lectures 6 and 7, retain one more term beyond the linear one in the Maclaurin expansion of $\sin\theta$:

$$\ddot{\varphi} = -\omega_0^2\varphi + \frac{\omega_0^2}{6}\varphi^3 \,. \tag{8.2}$$

Since $\varphi \ll 1$, we consider the last term in (8.2) as a small perturbation. As in Lecture 3, we expect that such a perturbation will cause only a small correction to the soltion $\varphi^{(0)}$ of the linear (harmonic oscillator) model,

$$\ddot{\varphi^{(0)}} = -\omega_0^2\varphi^{(0)} \,. \tag{8.3}$$

The general solution of (8.3) is (see (6.14e) of Lecture 6):

$$\varphi^{(0)} = A\cos(\omega_0 t + \text{constant phase}) \,.$$

We can set the phase to zero (as explained at the beginning of Sec. 6.9, this corresponds to a mere shift of the initial time). Moreover, since φ, and hence $\varphi^{(0)}$, is small, then so is the amplitude A. Thus,

$$\varphi^{(0)} = A\cos(\omega_0 t) = \frac{A}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t}), \qquad A \ll 1. \tag{8.4}$$

Let us look for the solution of (8.2) in the form:

$$\varphi = \varphi^{(0)} + \varphi^{(1)}, \qquad |\varphi^{(1)}| \ll |\varphi^{(0)}| \,. \tag{8.5}$$

We now need to determine how small $\varphi^{(1)}$ is compared to $\varphi^{(0)}$. To this end, substitute (8.5) into (8.2):

$$\underline{\ddot{\varphi^{(0)}}} + \ddot{\varphi^{(1)}} = -\underline{\omega_0^2\varphi^{(0)}} - \omega_0^2\varphi^{(1)} + \frac{\omega_0^2}{6}(\varphi^{(0)} + \varphi^{(1)})^3 \,.$$

The underlined (i.e., main-order) terms cancel by virtue of (8.3), and the rest of the above equation is:

$$\ddot{\varphi^{(1)}} = -\omega_0^2\varphi^{(1)} + \frac{\omega_0^2}{6}[\varphi^{(0)\,3} + 3\varphi^{(0)\,2}\varphi^{(1)} + \ldots]\,.$$

Recall that $\varphi^{(0)} = O(A)$. Then, matching the orders of magnitude of the term on the l.h.s. and the first two terms on the r.h.s., we conclude that $\varphi^{(1)} = O(A^3)$. These three terms are then all of order $O(A^3)$. Then the last term on the r.h.s. is $O(A^2 \cdot A^3) \ll O(A^3)$ and hence can be dropped. Thus, $\varphi^{(1)}$ satisfies:

$$\ddot{\varphi^{(1)}} = -\omega_0^2\varphi^{(1)} + \frac{\omega_0^2}{6} \cdot A^3\cos^3(\omega_0 t). \tag{8.6}$$

This is an equation of a harmonic oscillator with an external force. To apply to it the results of Lecture 6, we need to represent $\cos^3(\omega_0 t)$ as some linear combination of cosines (and/or sines) of some frequencies. This can be done by looking up an appropriate identity for $\cos^3 x$. We, however, will *derive* such an identity starting with the complex representation of $\cos x$:

$$\begin{aligned}
\cos^3 x &= \left(\frac{1}{2}\left(e^{ix} + e^{-ix}\right)\right)^3 \\
&= \frac{1}{8}(\underline{(e^{ix})^3} + \underline{\underline{3(e^{ix})^2 e^{-ix}}} + \underline{\underline{3e^{ix}\cdot(e^{-ix})^2}} + \underline{(e^{-ix})^3}) \\
&= \frac{1}{8}(\underline{e^{3ix}} + \underline{e^{-3ix}} + 3[e^{ix} + e^{-ix}]) \\
&= \frac{1}{4}\left(\frac{1}{2}(e^{3ix} + e^{-3ix}) + \frac{3}{2}(e^{ix} + e^{-ix})\right) \\
&= \frac{1}{4}(\cos 3x + 3\cos x)\,.
\end{aligned} \tag{8.7}$$

Thus, (8.6) can be rewritten as

$$\ddot{\varphi^{(1)}} = -\omega_0^2\varphi^{(1)} + \frac{\omega_0^2}{24} \cdot A^3(\cos(3\omega_0 t) + 3\cos(\omega_0 t)) \equiv -\omega_0^2\varphi^{(1)} + F_1(t) + F_2(t)\,. \tag{8.8}$$

That is,

$$F_1(t) = \frac{\omega_0^2}{24}A^3\,\cos(3\omega_0 t), \qquad F_2(t) = \frac{\omega_0^2}{8}A^3\,\cos(\omega_0 t).$$

Note that the frequencies of $F_1(t)$ and $F_2(t)$ are $3\omega_0$ and ω_0, respectively, while the natural frequency of the harmonic oscillator (8.3) is ω_0.

The general solution of the linear inhomogeneous equation (8.8) can be found using an *extended principle of linear superposition* (valid *only* for linear equations). It states:

$$\varphi^{(1)} = \begin{array}{c}\text{homogeneous}\\ \text{solution}\end{array} + \begin{array}{c}\text{particular}\\ \text{solution}\\ \text{caused by}\\ F_1(t)\end{array} + \begin{array}{c}\text{particular}\\ \text{solution}\\ \text{caused by}\\ F_2(t)\end{array}. \tag{8.9}$$

We know the homogeneous solution of (8.8):

$$\varphi^{(1)}_{\text{hom}} = c_1\,\cos\omega_0 t + c_2 \sin\omega_0 t, \qquad c_1,\, c_2 = O(A^3)\,. \tag{8.10a}$$

The particular solution caused by $F_1(t)$ is found from Eq. (6.21) of Lecture 6:

$$\varphi_1^{(1)} = \frac{\frac{\omega_0^2}{24}A^3}{\omega_0^2 - (3\omega_0)^2}\cos(3\omega_0 t)\,. \qquad (8.10b)$$

Equation (6.21) was written for $\omega_0 = 1$ but can be straightforwardly generalized to the case $\omega_0 \neq 1$. So, verify that (8.10b) reduces to (6.21) (for an appropriate F_0) when $\omega_0 = 1$.

Finally, the particular solution caused by $F_2(t)$ is found from Eq. (6.24) of Lecture 6, which is similarly generalized to account for $\omega_0 \neq 1$:

$$\varphi_2^{(1)} = \frac{\frac{\omega_0^2}{8}A^3}{2}\cdot\frac{1}{\omega_0^2}(\omega_0 t)\cdot\sin(\omega_0 t)\,. \qquad (8.10c)$$

Note that force $F_2(t)$ has a frequency ω_0, which equals the natural frequency of the unperturbed oscillator; see Eq. (8.3). So, as explained in Sec. 6.9 of Lecture 6, such a force causes *resonance*, i.e. a linear, unbounded growth of the amplitude of the oscillations in time. This is the *main conclusions* that one should draw from solution (8.10c).

Substituting expressions (8.10) into (8.9), we have:

$$\begin{array}{rcll}
\varphi^{(1)} & = & c_1\cdot\cos\omega_0 t + c_2\cdot\sin\omega_0 t & \Big|\; = O(A^3) \ll O(A) \text{ for all } t \\
 & - & \dfrac{A^3}{192}\cos(3\omega_0 t) & \Big|\; = O(A^3) \ll O(A) \text{ for all } t \\
 & + & \dfrac{A^3}{16}(\omega_0 t)\sin(\omega_0 t)\,. & \Big|\; = O(A^3\cdot(\omega_0 t)) \ll O(A) \ \textbf{ONLY} \text{ for } (\omega_0 t) \ll \frac{1}{A^2}
\end{array} \qquad (8.11)$$

For $(\omega_0 t) \geq \frac{1}{A^2}$ ($\gg 1$), i.e. for sufficiently large times, the last term in (8.11) becomes $O(A^3 \cdot \frac{1}{A^2}) = O(A)$ or greater. That is, the presumably small correction $\varphi^{(1)}$ becomes of the order of, or greater than, the principal term $\varphi^{(0)}$, and hence the main assumption, $|\varphi^{(1)}| \ll |\varphi^{(0)}|$, of expansion (8.5) becomes violated. And then both (8.5) and hence (8.11) are no longer valid.

To summarize, we have solved Eqs. (8.2) and (8.3), which differ only by a small term φ^3. We have expected that their solutions would also differ by a small amount. While this is indeed so for times t of order one, it is *not* so for sufficiently large times. Specifically, the two solutions differ significantly for $t = O(1/A^2) \gg 1$. This may sound counterintuitive: how can a perturbation that is small at all times cause large changes to a stable system?

Let us point out a simple analogy that answers this question. Consider some abstract quantity x that satisfies an equation

$$\dot{x} = 0\,. \qquad (8.12)$$

This is a stable system with the solution $x = \text{const}$. Now consider a slightly perturbed version of (8.12):

$$\dot{x} = \epsilon, \qquad \epsilon \ll 1. \qquad (8.13)$$

Its (exact) solution is

$$x = \text{const} + \epsilon\, t \equiv x^{(0)} + x^{(1)}\,. \qquad (8.14)$$

Obviously, $x^{(1)} \ll x^{(0)}$ only as long as $\epsilon t \ll \text{const}\,(= O(1))$. For $t \geq (1/\epsilon) \gg 1$, the correction caused by the perturbation becomes larger than the principal solution. Since solution (8.14) is exact, this is a true phenomenon rather than a result of an approximation. What happens in this case is that the effect of the small pertrbation (the r.h.s. of (8.13)) accumulates and causes large changes to the solution over a long time.

8.2 The reason behind the failure of a regular perturbation theory for (8.2), and its physical implications

We have seen that the way in which a small perturbation causes a large change is by accumulating over long time. But we still would like to understand *in what sense* the solution of (8.2) is substantially different from the solution of (8.3). Namely, what does it mean, physically, that the last term in (8.11) grows in time? If this is actually so, it would be very counterintuitive, because there is no external force acting on the apparently stable system (8.2) that could cause its solution to grow.

To uncover the reason, let us look back at Eq. (8.8). It has two perturbation terms. The term that causes the growth of $\varphi^{(1)}$ is proportional to $\cos(\omega_0 t)$, i.e. to $\varphi^{(0)}$. The same term would occur also if we consider an equation

$$\ddot{\varphi} = -\omega_0^2 \varphi + \epsilon \omega_0^2 \varphi, \qquad \epsilon \ll 1, \tag{8.15}$$

and seek its solution in the form (8.5), where $\varphi^{(1)} = O(\epsilon)$. Thus, instead of considering Eq. (8.8), we will now consider the simpler Eq. (8.15). Then, we will compare the solutions of (8.8) and Eq. (8.15) and from the simpler solution of (8.15) will identify the recipe of how to "fix" the problematic, growing solution (8.10c) of Eq. (8.8).

Substituting (8.5) in (8.15) one has:

$$\underline{\ddot{\varphi}^{(0)}} + \ddot{\varphi}^{(1)} = \underline{-\omega_0^2 \varphi^{(0)}} - \omega_0^2 \varphi^{(1)} + \epsilon \omega_0^2 \varphi^{(0)} + \epsilon \omega_0^2 \varphi^{(1)} \,.$$

The underlined terms cancel in view of (8.3), and the last term on the r.h.s. should be ommitted as being $O(\epsilon^2)$. The remaining $O(\epsilon)$-terms are:

$$\begin{aligned} \ddot{\varphi}^{(1)} &= -\omega_0^2 \varphi^{(1)} + \epsilon \omega_0^2 \varphi^{(0)} \\ &= -\omega_0^2 \varphi^{(1)} + \epsilon \omega_0^2 \cdot A \cos(\omega_0 t) \,. \end{aligned} \tag{8.16}$$

This is the equation of a harmonic oscillator acted upon by a resonant force, considered in Sec. 6.9 of Lecture 6. Its solution is written down similarly to (8.10c):

$$\varphi^{(1)} = \begin{matrix} \text{homogeneous} \\ \text{solution} \end{matrix} + \frac{\epsilon \omega_0^2 A}{2} \cdot \frac{(\omega_0 t)\sin(\omega_0 t)}{\omega_0^2} \,. \tag{8.17}$$

Thus, the resonant growth of the correction to the main-order solution occurs also for model (8.15). But this model can be solved exactly once we notice that it can be written as

$$\begin{aligned} \ddot{\varphi} &= -\omega_0^2 (1-\epsilon)\varphi \qquad \Leftrightarrow \\ \ddot{\varphi} &= -(\omega_0 \sqrt{1-\epsilon})^2 \varphi \,. \end{aligned} \tag{8.18}$$

The *exact* solution of (8.18), and hence of (8.15), is also an oscillation with frequency

$$\omega = \omega_0 \sqrt{1-\epsilon} = \omega_0 \left(1 - \frac{\epsilon}{2} + O(\epsilon^2)\right) : \tag{8.19}$$

$$\begin{aligned} \varphi &= A\cos(\omega t) = A\cos\left(\omega_0 t - \frac{\epsilon \omega_0}{2} t + O(\epsilon^2)\right) \\ &= A\cos\omega_0 t + A\frac{\epsilon\omega_0}{2} t \cdot \sin(\omega_0 t) + O(\epsilon^2) \,. \end{aligned} \tag{8.20}$$

In the last line, we have used the first two terms of the Taylor expansion for $\cos(x + \Delta x)$ (verify). Note that the $O(\epsilon)$ term in (8.20) is precisely the last term in (8.17), as it should be. Of course, for large times when $\epsilon\omega_0 t \ge 1$, the Taylor series used in expanding the cosine in (8.20) is not valid. Note that Eq. (8.17) is not valid under the same condition $\epsilon\omega_0 t \ge 1$, because then $\varphi^{(1)} \not\ll \varphi^{(0)}$.

The above comparison of the exact solution, given by the first line of (8.20), and the approximate solution found from (8.17) (and confirmed by the second line of (8.20)) suggests *how* the problem of the growing term in $\varphi^{(1)}$ can be fixed:

One needs to include this term into the main-order solution with a slightly different frequency!

Mathematically, this can be done by reading Eq. (8.20) backwards (i.e. starting from the second line). That is:

$$\begin{aligned} A\cos\omega_0 t + A\frac{\epsilon\omega_0}{2}t\cdot\sin(\omega_0 t) &= \\ A\cos\left(\omega_0 t - \frac{\epsilon\omega_0}{2}t\right) + O(\epsilon^2) &\equiv \\ A\cos\left(\omega_0\left(1-\frac{\epsilon}{2}\right)t\right) + O(\epsilon^2). & \qquad \text{(8.20-backwards)} \end{aligned}$$

Returning now to our original problem, the perturbation expansion (8.5) for Eq. (8.2), we obtain (see (8.11)):

$$\begin{aligned} \varphi^{(0)} + \varphi^{(1)} &= \underbrace{A\cos(\omega_0 t)}_{\varphi^{(0)}} + \underbrace{(c_1\cos\omega_0 t + c_2\sin\omega_0 t)}_{\substack{\text{not essential; will only slightly change} \\ \text{the amplitude and phase of } \varphi^{(0)}}} \\ &- \underbrace{\frac{A^3}{192}\cos(3\omega_0 t)}_{\substack{\text{nongrowing} \\ \text{term}}} + \underbrace{\frac{A^3}{16}(\omega_0 t)\sin(\omega_0 t)}_{\substack{\text{combine this with } \varphi^{(0)} \\ \text{as in (8.20-backwards)}}} \\ &= A\left[\cos(\omega_0 t) - \left(-\frac{A^2}{16}(\omega_0 t)\right)\sin(\omega_0 t)\right] + (\text{nongrowing terms}) \\ &= A\cos\left(\omega_0 t - \frac{A^2}{16}\omega_0 t\right) + O\left(A\cdot(A^2)^2\right) + (\text{nongrowing terms}) \\ &= A\cos\left(\omega_0\left[1-\frac{A^2}{16}\right]t\right) + O(A^5) + (\text{nongrowing terms}). \end{aligned} \tag{8.21}$$

Omitting the term $(c_1\cos\omega_0 t + c_2\sin\omega_0 t)$, as pointed out above, we present the solution of (8.2) that is *valid for all times*:

$$\varphi = A\cos\left(\omega_0\left[1-\frac{A^2}{16}\right]t\right) - \frac{A^3}{192}\cos(3\omega_0 t) + O(A^5). \tag{8.22}$$

<u>**The main result**</u> of this and the previous sections can now be summarized as follows.

If the correction $\varphi^{(1)}$ produced by the regular perturbation theory for a harmonic oscillator contains linearly growing terms, then such terms can be included into the main-order solution with a slightly modified frequency. The remaining terms in the correction $\varphi^{(1)}$ are small for all times, and hence valid.

In connection with the above, let us mention two pieces of notation:

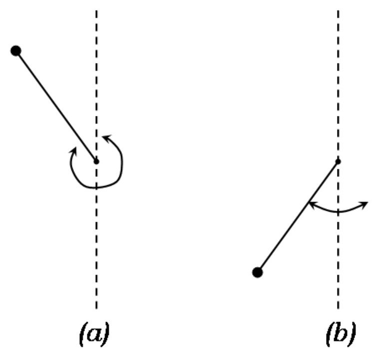
- Such a perturbation theory is called ***singular***. Here we had just a glimpse at one of its important ideas. The details are studied in a graduate course on the perturbation theory.
- The linear growth, like the one exhibited by the last terms in (8.11), (8.17), and (8.20), is referred to as ***secular*** growth. The perturbation terms that cause secular growth are, correspondingly, called ***secular terms*** (these are the last terms in (8.8) and (8.16)).

Let us also list *the effects* that the small cubic correction in the equation of motion has on the harmonic oscillations, as evidenced by (8.22):

1. The oscillation frequency changes (see the discussion about Eq. (8.23) below);
2. Small "ripple" whose frequency is three times as high as the original frequency, is produced. Such a ripple is often referred to as the ***third harmoinic***.

To conclude this section, we stress that (8.22) predicts that the frequency of oscillations of a pendulum will *decrease* if their amplitude will increase:

$$\omega = \omega_0 \left[1 - \frac{A^2}{16} \right] . \tag{8.23}$$



This makes sense since we know that the period of the oscillations in case (a) shown on the left is greater than the period in case (b). (Indeed, the amplitude of the oscillations tends to π, the period tends to infinity — it will take an ideal pendulum forever to leave its equilibrium state $\theta = \pi$.) Recall, however, that the period (and frequency) of very small oscillations is almost independent of their amplitude and equals $2\pi/\omega_0 = 2\pi/\sqrt{g/l}$.

Further intuitive exploration of the dependence of the frequency of nonlinear oscillations on their amplitude will be carried out in a homework problem.

8.3 Effect of a quadratic perturbation on the harmonic oscillator

We have found that the cubic terms would modify the frequency of the oscillations of a pendulum, as well as cause small corrections at a new frequency, $3\omega_0$. What will happen if the perturbation terms are quadratic in φ rather than cubic?

We will first answer this question mathematically, without relating it to a specific physical model. Then, in the next section, we will consider a model where the first correction to the harmonic oscillator model is a quadratic term.

Consider

$$\ddot{\varphi} = -\omega_0^2 \varphi + \epsilon \omega_0^2 \varphi^2, \qquad \epsilon \ll 1 \tag{8.24}$$

and, as before, seek a solution in the form (8.5) with $\varphi^{(1)} = O(\epsilon)$ and $\varphi^{(0)} = \cos(\omega_0 t)$. Substituting this into (8.24), we obtain:

$$\underline{\ddot{\varphi}^{(0)}} + \ddot{\varphi}^{(1)} = \underline{-\omega_0^2 \varphi^{(0)}} - \omega_0^2 \varphi^{(1)} + \epsilon \omega_0^2 (\varphi^{(0)} + \varphi^{(1)})^2 .$$

As before, the underlined terms cancel. Omitting $O(\epsilon^2)$-terms in the resulting equation yields:

$$\ddot{\varphi^{(1)}} = -\omega_0^2 \varphi^{(1)} + \epsilon\omega_0^2 \left(\varphi^{(0)}\right)^2 . \tag{8.25}$$

We now expand the last term in (8.25) similarly to how we expanded the last term of (8.6) in calculation (8.7):

$$\begin{aligned} \cos^2 x &= \left(\frac{1}{2}(e^{ix} + e^{-ix})\right)^2 \\ &= \frac{1}{4}(e^{2ix} + 2e^{ix}\cdot e^{-ix} + e^{-2ix}) \\ &= \frac{1}{2}\left(\frac{1}{2}(e^{2ix} + e^{-2ix}) + 1\right) \\ &= \frac{1}{2}(\cos(2x) + 1) \\ &= \frac{1}{2}\big(\cos(2x) + \cos(0x)\big) . \end{aligned} \tag{8.26}$$

The reason why we rewrote "1" as "$\cos(0x)$" will become clear shortly.

Next, we use (8.26) and (8.4) in the last term of (8.25) to obtain:

$$\ddot{\varphi^{(1)}} = -\omega_0^2 \varphi^{(1)} + \frac{\epsilon\omega_0^2}{2}\big(\cos(2\omega_0 t) + \cos(0t)\big) . \tag{8.27}$$

This equation is a counterpart of (8.8). It shows that the external force that arose from term $\left(\varphi^{(0)}\right)^2$ has two frequencies: $2\omega_0$ and 0. Then, using Eq. (6.21) of Lecture 6, we find the general solution of (8.27):

$$\varphi^{(1)} = \begin{array}{c}\text{homogeneous}\\ \text{solution}\end{array} + \frac{\epsilon\omega_0^2/2}{\omega_0^2 - (2\omega_0)^2}\cos(2\omega_0 t) + \frac{\epsilon\omega_0^2/2}{\omega_0^2 - 0^2}\cdot 1. \tag{8.28}$$

It is a counterpart of Eq. (8.10b).

Combining the results of these calculations and omitting the homogeneous solution in (8.28) as nonessential (because it is proportional to $\varphi^{(0)}$ and hence could be included into the latter), we have:

$$\varphi = \cos(\omega_0 t) - \frac{\epsilon}{6}\cos(2\omega_0 t) + \frac{\epsilon}{2} + O(\epsilon^2) . \tag{8.29}$$

Both $O(\epsilon)$-terms in (8.29), representing corrections to the main-order solution, remain small for all times. Thus, *quadratic corrections* to the harmonic oscillator model do *not* modify the oscillation freqency and cause *only* small, *non-secular*[10] effects.

Similarly to what we did upon studying the effect of a cubic correction, let us now list the effects that a small quadratic term in the equation of motion has on the harmonic oscillations, as per (8.29):

1. The average value of the oscillation becomes nonzero; see the term $(\epsilon/2)$. This means that the equilibrium value of the solution is slightly shifted.

2. Small "ripple" whose frequency is two times as high as the original frequency, is produced. Such a ripple is often referred to as the ***second harmoinic***.

[10] The term 'secular' was defined before Eq. (8.23).

8.4 Specific model with quadratic corrections

Recall the model of a pendulum on a rotating platform studied in Lecture 7; the corresponding equation of motion is (7.15). Consider its nonvertical (i.e., with $\theta_E \neq 0$ or π) equilibrium satisfying

$$R^2 \cos\theta_E - 1 = 0\,. \tag{8.30}$$

Near this equilibrium,

$$\theta = \theta_E + \varphi, \qquad \varphi \ll 1.$$

We need to substitute this into Eq. (7.15) of Lecture 7 and retain all terms of the orders $O(\varphi)$ and $O(\varphi^2)$. As auxiliary expansions, we need:

$$\begin{aligned} \sin\theta &= \sin(\theta_E+\varphi) &= \sin\theta_E + \frac{\cos\theta_E}{1!}\varphi + \frac{-\sin\theta_E}{2!}\varphi^2 + \ldots \\ \cos\theta &= \cos(\theta_E+\varphi) &= \cos\theta_E + \frac{-\sin\theta_E}{1!}\varphi + \frac{-\cos\theta_E}{2!}\varphi^2 + \ldots \end{aligned} \tag{8.31}$$

We substitute expansions (8.31) into Eq. (7.15) of Lecture 7 and, using shorthand notations

$$s \equiv \sin\theta_E, \qquad c \equiv \cos\theta_E\,,$$

obtain (verify):

$$\begin{aligned} \ddot\varphi &= \omega_0^2 \left(s + c\,\varphi - \frac{s}{2}\varphi^2\right)\left(R^2\left[c - s\,\varphi - \frac{c}{2}\varphi^2\right] - 1\right) \\ &= \omega_0^2 \left(s + c\,\varphi - \frac{s}{2}\varphi^2\right)\cdot R^2 \cdot \left(-s\,\varphi - \frac{c}{2}\varphi^2\right) \\ &= \omega_0^2 R^2 \left(-s^2\varphi - \frac{s\cdot c}{2}\varphi^2 - c\cdot s\cdot \varphi^2 + O(\varphi^3)\right)\,. \end{aligned}$$

Finally, omitting the higher-order terms, we have:

$$\ddot\varphi = -(\omega_0^2 R^2 \sin^2\theta_E)\cdot\varphi - \frac{3}{2}(\omega_0^2 R^2 \sin\theta_E\cos\theta_E)\varphi^2\,. \tag{8.32}$$

Thus the first-order corrections to the harmonic oscillator model obtained by the linearization near this nonvertical equilibrium, are quadratic. One can verify that for the other two, vertical, equilibria of this model (i.e., $\theta_E = 0$ and $\theta_E = \pi$), the first-order corrections are cubic.