

9 Continuous limits of difference balance equations

The purpose of this lecture is twofold. First, we will revisit the topic of setting up balance equations, which we considered in Lecture 5. This time our focus will be on how that approach can be used to derive continuous models starting from models with a finite number of “elements”. Moreover, some of the results derived here (that are independent of the continuous approximation) will be used in the next Lecture, where we will consider the *inverse problem* of identifying a parameter of a model from the model’s behavior (i.e., from the solution). Thus, preparing a background for Lecture 10 is the second purpose of this Lecture.

9.1 Newton’s law of cooling and the Heat equation

The temperature of a body that is not in thermal equilibrium with its surroundings changes in time: it decreases if the body is hotter than the environment and increases otherwise. Let u denote the temperature of the body in question and u_E denote the temperature of the environment. The “cooling law” whose idea was first formulated by Newton in 1701 states that

$$\frac{du}{dt} = \alpha (u_E - u), \quad (9.1a)$$

where α is a heat transfer coefficient (which depends on the dimensions of the body). This is a *linear inhomogeneous differential equation* for $u(t)$, and hence can be solved given α , u_E , and the initial condition

$$u(t = 0) = u_0. \quad (9.1b)$$

Note that when $u_E = \text{const}$ (i.e., the ambient temperature does not vary with time), Eq. (9.1a) is also a *separable differential equation*. In this case its solution is, as you will derive in the homework:

$$u(t) = u_E + (u_0 - u_E) e^{-\int_0^t \alpha(t') dt'}. \quad (9.2)$$

This solution predicts that the difference between the temperature of the body and the environment decays with time, so that $\lim_{t \rightarrow \infty} u(t) = u_E$.

Two comments are in order about solution (9.2). First, in the last term we explicitly indicated that the heat transfer coefficient, α , can depend on time. This assumption is *not* important in this section and, in fact, in all sections of this Lecture except the last one. You may proceed until Section 9.4 assuming that α and its counterpart r in Section 9.2 are constant. It is only in Section 9.4 that we will explore the effect of time-varying $\alpha(t)$ or $r(t)$.

The second comment is about the cosmetic appearance of the last term on the r.h.s. of Eq. (9.2). You may notice that it is inconvenient to have a somewhat complicated-looking expression in the exponent of the e . To address this, the scientific community has introduced notation ‘exp’ (for the ‘exponential’):

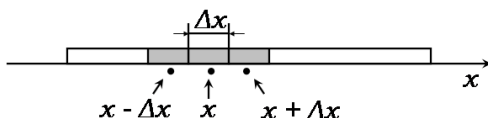
$$e^{f(t)} \equiv \exp[f(t)].$$

With this cosmetic change, Eq. (9.2) can be rewritten as:

$$u(t) = u_E + (u_0 - u_E) \exp \left[- \int_0^t \alpha(t') dt' \right]. \quad (9.2')$$

The Newton’s law of cooling is *local* in the sense that it ignores the fact that the temperature may (and in many cases does) depend on *where* in the body it is measured. (For example, the

temperature inside a human body is usually different than the skin temperature.) The theory that governs the distribution of temperature in an extended body was developed by Joseph Fourier at the beginning of the 19th century. A standard derivation, based on Fourier's ideas, of the equation for temperature distribution is found in Sec. 4.5 of the book by C. Groetsch. Here we will give a slightly different derivation based on the Newton's cooling law.



Consider a one-dimensional rod where the temperature is distributed along x . Consider three consecutive elements of length Δx in this rod, as shown on the left.

For the middle element we have:

$$\frac{du(x)}{dt} = \underbrace{\alpha(u(x - \Delta x) - u(x))}_{\text{from Newton's law at the left face}} + \underbrace{\alpha(u(x + \Delta x) - u(x))}_{\text{from Newton's law at the right face}}.$$

Equivalently,

$$\frac{du(x)}{dt} = \alpha(u(x + \Delta x) - 2u(x) + u(x - \Delta x)). \quad (9.3)$$

In (9.3) and the previous equation, we imply that the u 's also depend on t .

Now let us tend $\Delta x \rightarrow 0$. Recall that α depends on the dimensions of the body (the element in this case). Intuitively, we should expect that as $\Delta x \rightarrow 0$, $\alpha \rightarrow \infty$, because a very small body will acquire the temperature of its surroundings almost instantaneously. But how exactly does α depend on Δx : like $\alpha = O(\frac{1}{\Delta x})$, or in some other way? We will answer this question by considering the limit of the r.h.s. of (9.3) as $\Delta x \rightarrow 0$ and requiring that it be finite and nonzero.

Taylor expansions of $u(x + \Delta x)$ and $u(x - \Delta x)$ are:

$$\begin{aligned} u(x + \Delta x) &= u(x) + u'(x)\Delta x + u''(x)\frac{\Delta x^2}{2} + u'''(x)\frac{\Delta x^3}{6} + O(\Delta x^4), \\ u(x - \Delta x) &= u(x) - u'(x)\Delta x + u''(x)\frac{\Delta x^2}{2} - u'''(x)\frac{\Delta x^3}{6} + O(\Delta x^4). \end{aligned}$$

Adding these two expressions and subtracting $2u(x)$, to match the r.h.s. of (9.3), we find:

$$\begin{aligned} u(x + \Delta x) - 2u(x) + u(x - \Delta x) &= \underline{2u(x)} + 2u''(x)\frac{\Delta x^2}{2} + O(\Delta x^4) - \underline{2u(x)} \\ &= u''(x) \cdot \Delta x^2 + O(\Delta x^4), \end{aligned} \quad (9.4)$$

where the underlined terms cancel out. Substituting (9.4) into (9.3), we have:

$$\frac{\partial u(x, t)}{\partial t} = (\alpha \cdot \Delta x^2) \frac{u''(x, t)\Delta x^2 + O(\Delta x^4)}{\Delta x^2},$$

where we have accounted for the fact that u depends on both x and t and hence used the notation of a partial derivative $\partial/\partial t$. Now, taking the limit $\Delta x \rightarrow 0$ and *requiring that*

$$\lim_{\Delta x \rightarrow 0} \alpha \cdot \Delta x^2 = a, \quad (9.5)$$

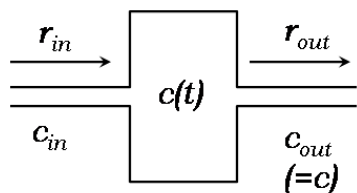
we find:

$$\boxed{\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}}. \quad (9.6)$$

This is the Heat equation, one of the classic equations of mathematical physics. It governs not only the evolution of temperature in a rod, but many other seemingly different physical processes.

Thus, to summarize this Section: We have derived a continuous model, Eq. (9.6), starting from the balance equation, Eq. (9.3), for a discrete model.

9.2 Mixing problems



Suppose that a tank contains solution of some substance whose concentration at moment t is $c(t)$. For simplicity, we will refer to this substance as ‘*salt*’ (without quotes). Suppose that solution with concentration c_{in} enters at a rate r_{in} , and the *well-mixed* solution exits at a rate r_{out} . Our first task in this Section will be to determine the concentration $c(t)$ as a function of time.

We begin by writing a balance equation for $V(t)$, the volume of water in the tank:

$$\frac{dV}{dt} = \left(\begin{array}{c} \text{Rate of flow of} \\ \text{water } \underline{\text{in}} \end{array} \right) - \left(\begin{array}{c} \text{Rate of flow of} \\ \text{water } \underline{\text{out}} \end{array} \right) = r_{\text{in}} - r_{\text{out}}. \quad (9.7)$$

Our next step is find an equation for the rate of change of the concentration of salt. The concentration is defined as

$$c(t) = \frac{q(t)}{V(t)}, \quad (9.8)$$

where $q(t)$ is the total amount of salt in the tank of volume V . Now, the **key observation** is that to determine the evolution of the concentration, one must *write the balance equation not for the concentration itself but for the amount of salt*. For the latter we have, similarly to (9.7):

$$\begin{aligned} \frac{dq}{dt} &= \left(\begin{array}{c} \text{Rate of flow} \\ \text{of salt } \underline{\text{in}} \end{array} \right) - \left(\begin{array}{c} \text{Rate of flow} \\ \text{of salt } \underline{\text{out}} \end{array} \right) \\ &= \left(\begin{array}{c} \text{Rate of flow} \\ \text{of water } \underline{\text{in}} \end{array} \right) \cdot \left(\begin{array}{c} \text{Amount of salt per vol.} \\ \text{of incoming water} \end{array} \right) - \left(\begin{array}{c} \text{Rate of flow} \\ \text{of salt } \underline{\text{out}} \end{array} \right) \left(\begin{array}{c} \text{Amount of salt per vol.} \\ \text{of outgoing water} \end{array} \right) \\ &= r_{\text{in}} \cdot c_{\text{in}} - r_{\text{out}} \cdot c_{\text{out}} \end{aligned} \quad (9.9)$$

Since in the well-mixed approximation, $c_{\text{out}} = c(t) = q(t)/V(t)$, then (9.9) yields:

$$\frac{dq}{dt} = r_{\text{in}} \cdot c_{\text{in}} - r_{\text{out}} \cdot \frac{q(t)}{V(t)}. \quad (9.10)$$

This is a linear inhomogeneous equation, which can be solved, e.g., by the method of variation of parameters. However, we will only consider the special case when

$$r_{\text{in}} = r_{\text{out}} \equiv r(t), \quad (9.11)$$

and hence from (9.7), one has $V(t) = \text{const}$. In this case Eq. (9.10) is also a *separable* differential equation (verify). Dividing both sides of (9.10) by $V = \text{const}$, we have:

$$\frac{dc}{dt} = \frac{r(t)}{V} (c_{\text{in}} - c). \quad (9.12a)$$

This equation has the same form as the Newton's cooling law, Eq. (9.1a). Therefore, its solution is, by analogy with Eq. (9.2):

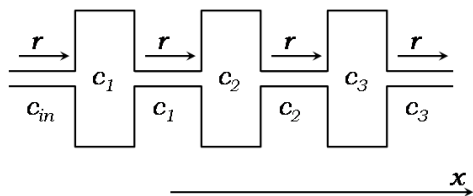
$$c(t) = c_{\text{in}} + (c_0 - c_{\text{in}}) \exp \left[-\frac{1}{V} \int_0^t r(t') dt' \right], \quad (9.13)$$

where the notation 'exp' was introduced after Eq. (9.2), and

$$c(t = 0) = c_0. \quad (9.12b)$$

Similarly to Eq. (9.2), Eq. (9.13) describes the concentration approaching the equilibrium value c_{in} as $t \rightarrow \infty$.

Now let us consider a situation where tanks are connected in sequence, and there is a unidirectional flow (say, from left to right):



Then in the well-mixing approximation, as before, the concentration of salt in the j -th tank satisfies:

$$\frac{dc_j}{dt} = \frac{r}{V} (c_{j-1} - c_j). \quad (9.14)$$

Let us consider the continuous limit, where the tanks of width $\Delta x \rightarrow 0$ are labeled with the value of the x -coordinate at which they are located (i.e., x is the continuous counterpart of j). Then, using $V = A\Delta x$, where A is the cross-sectional area of the tanks perpendicularly to the x -axis, we find from Eq. (9.14):

$$\frac{\partial c(x, t)}{\partial t} = \frac{r}{A} \cdot \frac{c(x - \Delta x, t) - c(x, t)}{\Delta x}. \quad (9.15)$$

Now, using the Taylor expansion for $c(x - \Delta x)$:

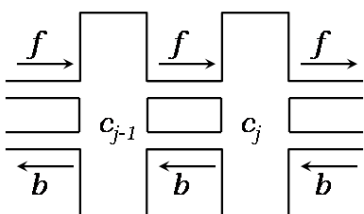
$$\lim_{\Delta x \rightarrow 0} \frac{c(x - \Delta x) - c(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[c(x) - c'(x)\Delta x + O(\Delta x^2)] - c(x)}{\Delta x} = -c'(x). \quad (9.16)$$

Together, Eqs. (9.15) and (9.16) yield:

$$\boxed{\frac{\partial c}{\partial t} + \frac{r}{A} \frac{\partial c}{\partial x} = 0}. \quad (9.17)$$

This is another classic equation of mathematical physics called the (one-dimensional) *Wave Equation*. Unlike the Heat Equation, which describes diffusion-like processes, the Wave equation *describes convection* (in this case, unidirectional).

Equation (9.17) may describe, in a certain approximation, a flow of some solution through a (one-dimensional) porous medium. In this case every element of width Δx of the medium is considered as a small tank, and the flow of the solution is considered to be unidirectional.



At home you will consider a more realistic model of a porous medium where the “tanks” are connected in such a way that the flow is allowed in both directions, as shown on the left. You will be asked to derive a generalization of Eq. (9.17) in this case.

9.3 Mortgage repayment

In the two previous sections, we started with models that were discrete in space but continuous in time and derived models that were continuous in both time and space. In this section, we will consider a model that is discrete in time and does *not contain* a space dimension. We will be interested in the continuous-time version of this model.

Suppose one deposits a sum of P_0 dollars in a bank account which compounds interest of $r \cdot 100\%$ every year. Then in one year the account will have

$$P_1 = P_0(1 + r), \quad (9.18a)$$

and in n years,

$$P_n = P_0(1 + r)^n. \quad (9.18b)$$

Now suppose that instead of being compounded once a year, as assumed in (9.18), the interest is compounded daily. Then:

$$\begin{aligned} P_{1 \text{ day}} &= P_0 \left(1 + \frac{r}{365}\right) \\ P_{2 \text{ days}} &= P_0 \left(1 + \frac{r}{365}\right)^2 \\ P_{1 \text{ year}} &= P_0 \left(1 + \frac{r}{365}\right)^{365} \\ P_{n \text{ years}} &= P_0 \left(1 + \frac{r}{365}\right)^{n \cdot 365}. \end{aligned}$$

Since $365 \gg 1$, we can approximately rewrite the last equation as:

$$P_{n \text{ years}} \approx P_0 \cdot \lim_{N \rightarrow \infty} \left(1 + \frac{r}{N}\right)^{n \cdot N}. \quad (9.19)$$

This is an indeterminate form of the type 1^∞ , and at home you will show that

$$\lim_{N \rightarrow \infty} \left(1 + \frac{r}{N}\right)^{n \cdot N} = e^{r \cdot n}. \quad (9.20)$$

For now we will only answer the following simple question: What is the *effective* annual percent rate (APR) compounded once per year that yields the same increment of the account as the nominal APR of $r \cdot 100\%$ compounded continuously (or daily)? To this end, note that

$$1 + r_{\text{eff}} = \left(1 + \frac{r}{365}\right)^{365} \approx e^r.$$

Thus, the answer to the above question is:

$$r_{\text{eff}} = e^r - 1. \quad (9.21)$$

To repeat, (9.21) relates the nominal APR of $r \cdot 100\%$ compounded continuously with the APR of $r_{\text{eff}} \cdot 100\%$ that provides the same yield when compounded once a year. From now on we will disregard the distinction between r and r_{eff} , arguing that if we know one of these rates, we can always find the other one from (9.21).

Now let us suppose that a person takes a mortgage for the principal sum (i.e., initial amount) of P_0 dollars and is obligated to repay it in M years. The person is charged a fixed interest of $(r/12) \cdot 100\%$ every month on the remaining balance, and pays a fixed amount of x dollars every month. What should this x be for the loan to be paid off in exactly M years?

The balance equation for the amount P_j that the person still owes after the j -th month is:

$$\underbrace{P_{j+1}}_{\substack{\text{total amount} \\ \text{owed after} \\ (j+1)\text{th month}}} = \underbrace{P_j}_{\substack{\text{total amount} \\ \text{owed after} \\ j\text{-th month}}} + \underbrace{\frac{r}{12}P_j}_{\substack{\text{the interest} \\ \text{on the} \\ \text{owed amount}}} - \underbrace{x}_{\substack{\text{monthly} \\ \text{payment}}}. \quad (9.22)$$

Although it is possible to solve this linear inhomogeneous *difference* equation (similarly to how it is possible to solve a linear inhomogeneous *differential* equation), the corresponding solution is rather unwieldy. Therefore, we will consider the continuous limit of Eq. (9.22) and then find *its* solution (which can be done much easier). This continuous limit follows with one intermediate step:

$$\frac{P_{j+1} - P_j}{1 \text{ month}} = \frac{r}{12}P_j - x, \quad \Rightarrow \quad \frac{dP}{dt} = \frac{r}{12}P - x = \left(-\frac{r}{12}\right) \left(\frac{x}{(r/12)} - P\right). \quad (9.23)$$

In (9.23), P is a function of the *continuous* variable t (measured *in months*). Note that the r.h.s. of (9.23) has been put in the form of the r.h.s.'s of Eqs. (9.1a) and (9.12a). Therefore, we can immediately write down its solution, replacing in Eq. (9.2) α with $(-r/12)$, u_E with $x/(r/12)$, and u_0 with P_0 :

$$P(t) = \frac{x}{(r/12)} + \left(P_0 - \frac{x}{r/12}\right) e^{\frac{r}{12}t} \quad (9.24)$$

(recall that we assumed the interest rate r to be constant). If the loan is paid off in $T = 12 \cdot M$ months, this means that

$$P(T) = 0. \quad (9.25)$$

Substituting (9.25) into (9.24) with $t = T$ and solving for x , we obtain (verify):

$$x = \frac{r}{12} \frac{P_0 e^{rM}}{e^{rM} - 1}. \quad (9.26)$$

This is the constant monthly payment.

9.4 Effect of rapid oscillations of the rate coefficient on the solution

Consider Eq. (9.12a), which determines the salt concentration in the tank when the incoming solution is pumped into the tank at the same rate with which the well-mixed solution is pumped out of the tank. Suppose that one has two pumping cases. The pumping rate in the first case is denoted as $r_1(t)$. In the second case, the pumping rate differs from r_1 by an oscillating term:

$$r_2(t) = r_1(t) + \Delta r \cdot \sin \omega t. \quad (9.27)$$

We will assume that the amplitude Δr of this “ripple” is *not* small; that is, it has the same order of magnitude as $r_1(t)$. However, we will assume that during the time of interest, t , the oscillations occur *many times*:

$$\omega t \gg 1. \quad (9.28)$$

This is schematically illustrated in the figure below. An *equivalent way* to say that “oscillations occur many times” is to say that they are *fast*. Indeed, fast oscillations occur many times during a given time interval. Therefore, *below we will refer to the ripple satisfying condition (9.28) as fast*. We want to find out how much the two resulting concentrations, $c_1(t)$ and $c_2(t)$, obtained from the respective solutions (9.13), differ from each other because of such a fast ripple.

The pumping rate $r(t)$ enters only into the exponent in (9.13) as a term integrated over time. Using (9.27), we can write:

$$\int_0^t r_2(t') dt' = \int_0^t r_1(t') dt' + \int_0^t \Delta r \cdot \sin(\omega t') dt'.$$

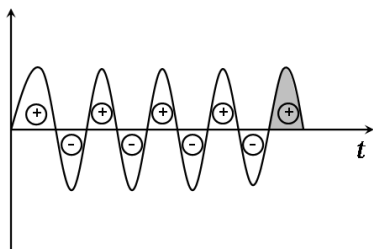
To estimate the *relative* contribution of the ripple to the difference of the two solutions $c_2(t)$ and $c_1(t)$ in (9.13), we assume that $r_1(t) \equiv r_1 = \text{const}$. This is not essential, but will simplify our consideration. Then, the first term on the r.h.s. of the above equation is simply

$$\int_0^t r_1 dt' = r_1 \cdot t. \quad (9.29)$$

The second term above is:

$$\begin{aligned} \Delta r \cdot \int_0^t \sin(\omega t') dt' &= \frac{\Delta r}{\omega} (1 - \cos(\omega t)) \\ &= \frac{\Delta r}{\omega} \cdot O(1) \\ &= \frac{\Delta r \cdot t}{\omega t} \cdot O(1) \\ &\stackrel{\text{see (9.28)}}{=} \underbrace{\Delta r \cdot t \cdot O\left(\frac{1}{\omega t}\right)}_{\ll 1} \stackrel{(9.29) \ \& \ \Delta r = O(r_1)}{=} O(r_1) \cdot t \cdot O\left(\frac{1}{\omega t}\right) \\ &\ll \int_0^t r_1 dt'. \end{aligned} \quad (9.30)$$

This calculation shows that the faster the ripple, the smaller its relative contribution to the integral in the exponent of (9.13) and hence the smaller its effect on the solution.



Visually, this can be explained as follows. The integral (i.e., the area) of each “positive” half-period is exactly cancelled by the negative area of the following half-period. Thus, if the number of whole periods is an integer, then the integral is exactly zero.

The integral deviates from zero when the contribution from the “positive” half-period is not exactly cancelled by that of the “negative” half-period. Therefore, the *maximum value* of the integral occurs when the number of whole periods is semi-integer. This maximum value is simply the area of the “positive” half-period and in the order-of-magnitude sense is:

$$\text{height} \cdot \text{width} = O(1) \cdot O\left(\frac{1}{\omega}\right) = O\left(\frac{1}{\omega}\right).$$

Thus, the main conclusion of this section is that high-frequency (i.e., fast) oscillations of the coefficient $r(t)$ do not significantly change the value of $\int_0^t r(t') dt'$. In the context of the problem considered in Sec. 9.2, this means that if the pumping rate $r(t)$ through a tank fluctuates rapidly about some mean value, this will only slightly affect the concentration $c(t)$ of the solution in the tank at any given time. Similarly, rapid fluctuations of the interest rate $r(t)$ about a mean value will only slightly affect the owed dollar amount $P(t)$ and hence the payments x , considered in Sec. 9.3.

In a homework problem you will extend the above conclusion of the effect of high-frequency oscillations of $t(t)$ to integrals of the form $\int_0^t f(t') r(t') dt'$, where $f(t)$ is any sufficiently slowly varying function. We will use this result in later Lectures.