Extra-Credit Mathematica Lab on Surfaces of Revolution

Dumbbells, Squished spheres, Hearts, and Surfaces with petals: From revolution to "generalized revolution"

Submission Instructions:

1. You may do any number of Exercises from this Bonus Lab. Note, however, that while Exercise 1 merely uses the method of the lecture on Sec. 16.6, Exercises 2–4 generalize that method in incremental steps.
2. Hand in your derivations of equations on paper. Include (brief, but sufficient) explanations.
3. E-mail me your Mathematica notebook with commands. Make sure to format it so that I can easily identify your commands with the Exercise that they are for.
4. The rest of the rules are the same as for other Extra-Credit assignments (*not* regular Labs!).

Goal of this Lab:

Surfaces related to surfaces of revolution describe many shapes occurring in Nature. In this Lab, you will learn how to extend the concept of revolution, which in the lecture on Sec. 16.6 is meant in the "usual" sense, i.e. along a circular path, to non-circular paths, and beyond.

1 Dumbbell (10%)

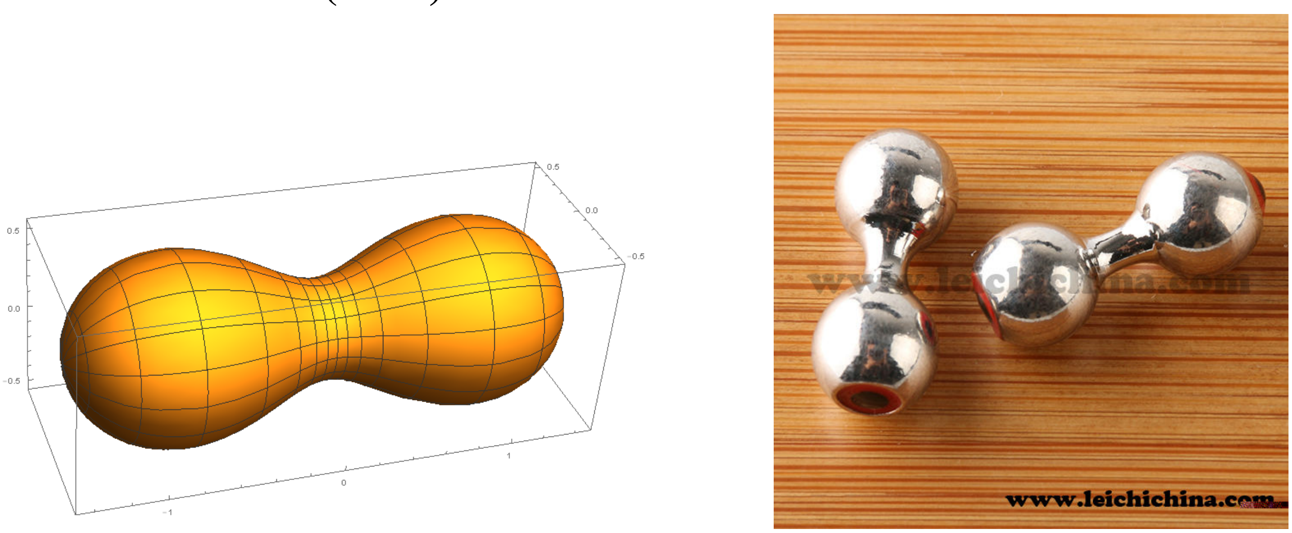


Figure 1: Left: The shape considered in this Exercise. Right: Dumbbells of similar shape.

Actually, most dumbbells are *not* of the shape shown in Fig. 1(left) and considered below, but some are: see Fig. 1(right). Atomic d-orbitals are of a similar shape, except that they, by laws of Quantum mechanics, have an infinitely thin waist. (Sometimes p-orbitals, too, are also drawn as this shape, although mathematically this is not quite correct.) Simple segments of "molecular surfaces", showing the distribution of the spatial electronic density in a molecule, may also have similar shapes.

Assignment:

1. Find the parametric equations of a peanut-shaped curve in the accompanying Mathematica notebook `EC_SurfOfRevol_ParamCurves.nb`. Experiment with parameters of those equations to choose their values that yield proportions that you like.

2. Using the material of the lecture on Sec. 16.6, write down the parametric equations of the surface shown in Fig. 1(left). Plot this surface in Mathematica. Make sure to use the ranges for u and v that create the surface only *once*. (E.g., ranges $[-10\pi,\ 10\pi]$ will probably still produce the desired surface, but smaller ranges will achieve the same purpose; you should use the smallest such ranges.)

2 "Squished" sphere (20%)

This object, scientifically called "spheroid" (see pictures in a Wikipedia article), approximates, among many other things, the shape of our Earth. It can be obtained by rotating half of an ellipse about the ellipse's axis that connects the two end points of that "half". This is how the surfaces shown in the aforementioned article have been generated. Indeed, the "original" half-ellipse was in the xz-plane (or maybe in the yz-plane) and was rotated around the z-axis. This can be stated based on the orientation of the parallels–meridians grid on the spheroid. The parametric equations of such a surface are very similar to those of a sphere, and they are obtained in exactly the same manner as the equations of a sphere are obtained in the corresponding example found in lecture on Sec. 16.6.

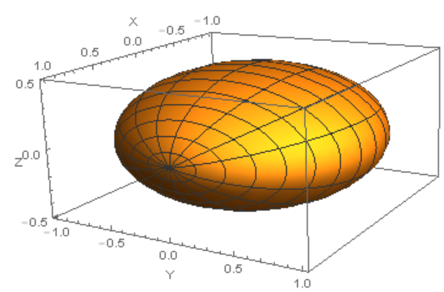
Assignment:

Generate the same surface (either oblate or prolate, your choice) in a *different* manner. Namely:

Start with a semi-circle in the xy-plane and move its points along ellipses (*not* circles, as in Sec. 16.6) around the x-axis.[a] This is how the surface in the figure on the right was generated: Note the orientation of the parallels–meridians grid on it.

Write down the parametric equations (with a brief explanation) and use them to plot your spheroid in Mathematica.

[a] Instead of saying 'move points along ellipses', I could have used the phrase 'elliptical rotation' to describe this motion; however, this is not a standard term in English (even though it is clear what it means).



3 Heart surface (30%)

The surface that you will be asked to create in this Exercise resembles that shown in the figure below on the left. Despite its name, it has little in common with the shape of the actual human heart (shown below on the right; click on the next link for similar images or on the following link for a Wikipedia article about the human heart). A story of how the stylized image of the heart came to existence can be found in this Wikipedia article.

The image in Fig. 2(left) is *not* a surface of revolution. Its equation in Cartesian coordinates can be found on a Wolfram MathWorld webpage. With much effort, one can obtain parametric equations

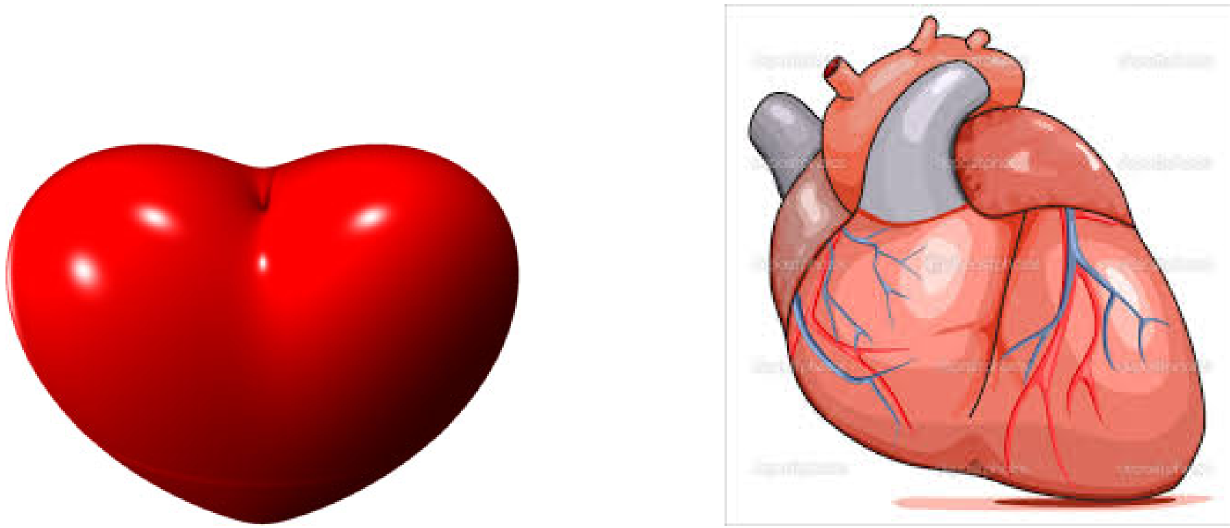


Figure 2: Left: A stylized heart symbol. Right: A more realistic heart image.

for this or similar-looking surface (either by playing with the Cartesian equations and using divine inspiration or by a more systematic method of generalizing, in a non-trivial way (see Section 5 below), the approach described in Exercise 2 above). We will, however, restrict ourselves to a more modest goal: generate a *surface of revolution* (in the generalized sense of Exercise 2) that resembles the surface in Fig. 2(left). Such a surface of revolution is shown in Fig. 3(left) below.

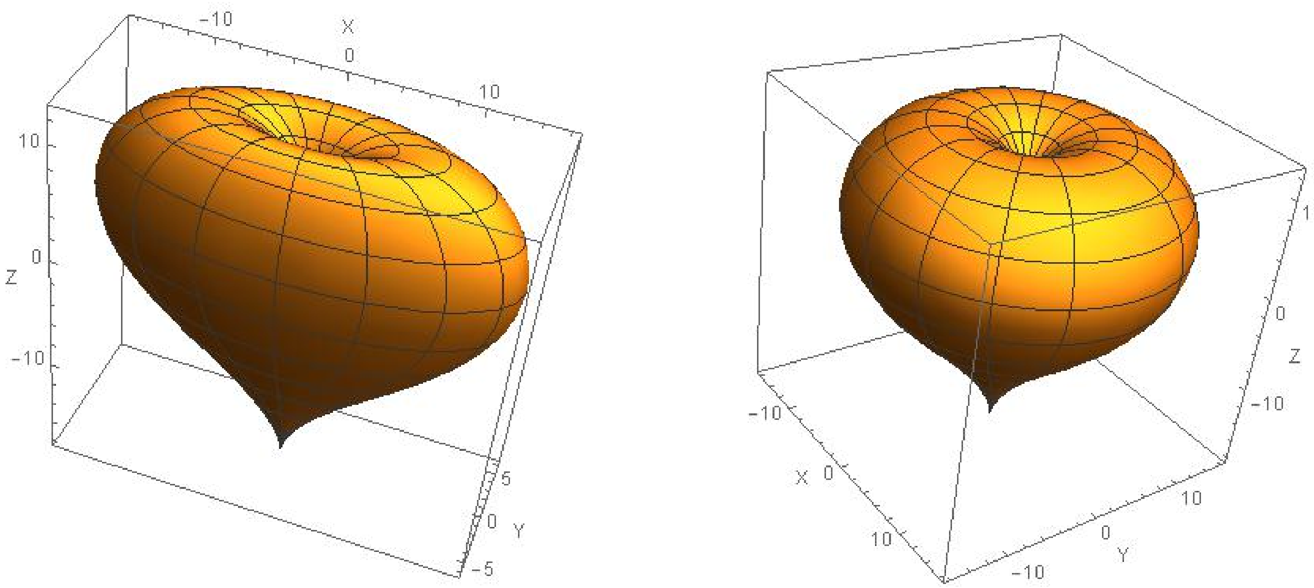


Figure 3: Left: A surface that you are to obtain in this Exercise. Right: A surface of revolution obtained from the Heart Curve by the method of the lecture on Sec. 16.6.

Assignment:

1. Find the parametric equations of a heart-shaped curve in the accompanying Mathematica notebook `EC_SurfOfRevol_ParamCurves.nb`.

2. Using the material of the lecture on Sec. 16.6, write down the parametric equations of the surface shown in Fig. 3(right). (This is correct: right, not left, yet.) Plot this surface in Mathematica. Make sure to use the ranges for u and v that create the surface only *once*.

3. Use the idea of Exercise 2 to "squish" that surface into something that looks like the surface in Fig. 3(left).

Note: You must do the "squishing" by manipulating the equations, not the aspect ratio of your axes.

4 Surface with petals (40%)

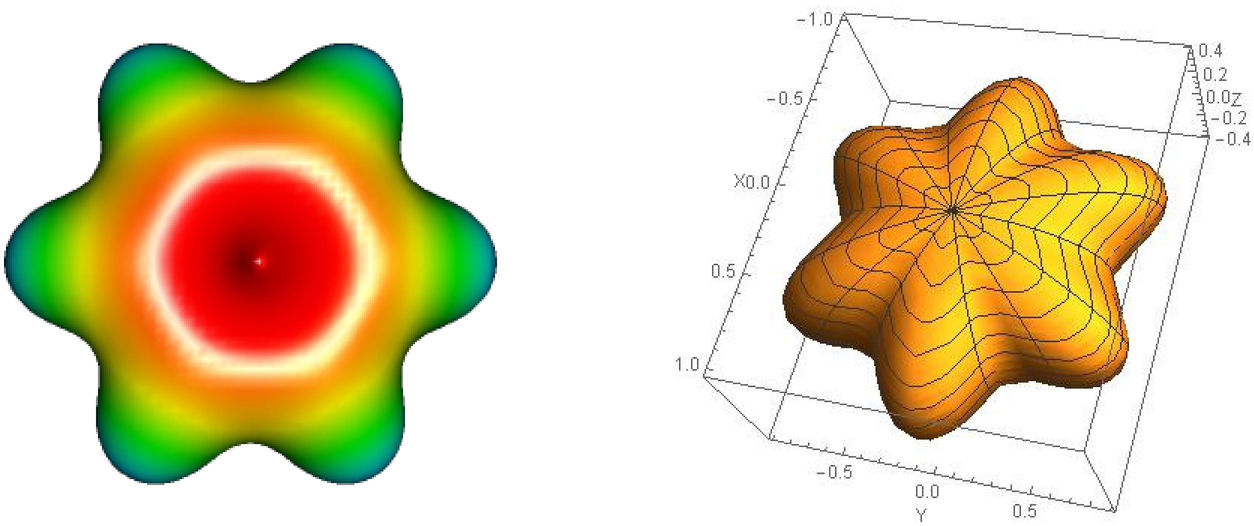


Figure 4: Left: Electrostatic potential surface in a benzene molecule. Right: A simplified version of the surface on the left.

In Exercise 1 you have seen a simplified image of the electronic density in a molecule, or the surface of constant electrostatic potential. Electronic densities of even small molecules have more complicated shapes. Such a shape for benzene (chemical formula C_6H_6), a chemical occurring in petroleum products and coal, is schematically shown in Fig. 4 above on the left. More information about, and images of, electrostatic potential surfaces ("maps") can be found online; e.g., see a tutorial by Dr. Kalju Kahn, from which the image in Fig. 4(left) was borrowed. In this Exercise you will be asked to produce a simplified version of this surface, shown in Fig. 4(right).

The idea by which such a seemingly non-rotational surface can be created is an extension of the idea of Exercise 2. There, we noticed that the "rotation" does not necessarily need to describe a *circular* path; the path could be an ellipse instead. You were asked to use the same idea in Exercise 3. But why restrict oneself to an ellipse? The path can be any parametric curve[1]! For example, it can be a curve shown in Fig. 5(left) below. Motion along such a cirve is not the usual circular rotation, but it can be thought of a "generalized rotation".

[1]This curve is to be closed if we want our surface to be smooth.

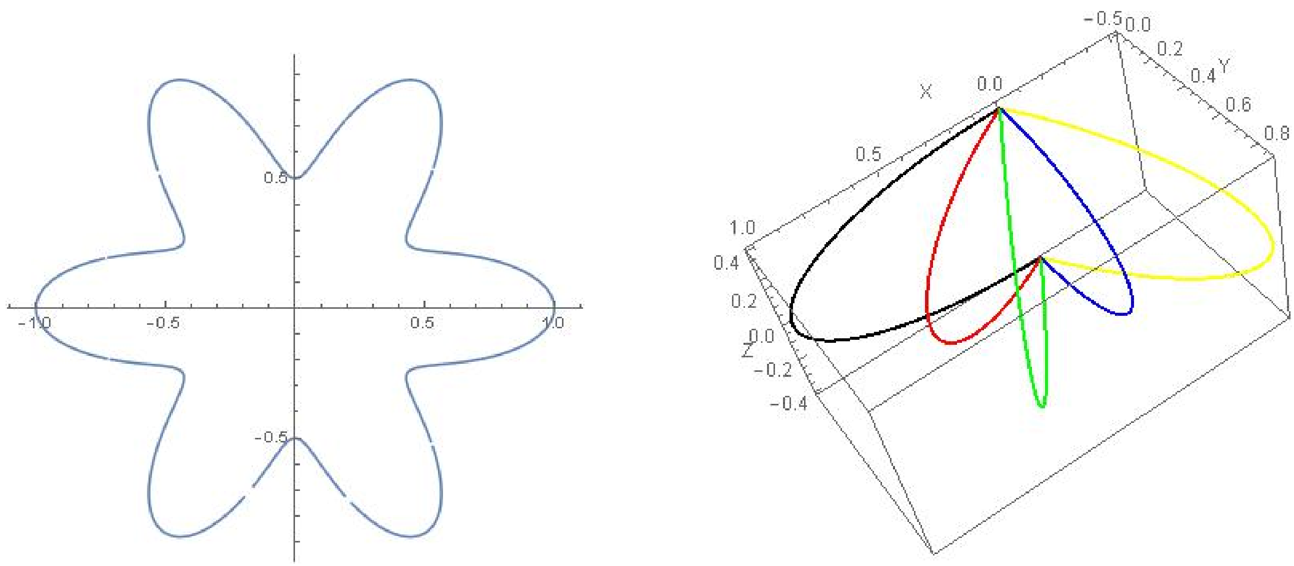


Figure 5: Left: A curve with 6 petals. Right: A schematics of how you can obtain the surface in Fig. 4(right) by "rotating" half-ellipses along the curve shown on the left. Black, Red, Green, Blue, and Yellow colors correspond to the angle of "generalized rotation" $\theta = 0,\ \pi/6,\ \pi/3,\ \pi/2$, and $2\pi/3$.

Assignment:

1. Find the parametric equations of a curve with petals in the accompanying Mathematica notebook `EC_SurfOfRevol_ParamCurves.nb`. Experiment with its parameters to create a curve similar to the one shown in Fig. 5(left).

2. Imagine a half-ellipse (say, oblate, i.e., "squished") in the xz-plane: it is depicted by the black curve in Fig. 5(right). Further imagine that its x-intercept is varied to trace the curve of Fig. 5(left) in the xy-plane. This is an example of the "generalized rotation" described in the paragraph before Fig. 5. A few consecutive locations of the so "rotated" ellipse are shown in Fig. 5(right).

Using this as a guide, write down parametric equations of the surface shown in Fig. 4(right). Plot this surface in Mathematica. Make sure to use the ranges for u and v that create the surface only *once*.

5 Further extensions of the "generalized rotation"

But why constrain oneself to force every point of the curve being rotated to follow the same trajectory? Why not make the trajectory being dependent on z (where the z-axis is assumed to be the axis of rotation, as in Exercises 3 and 4)? If one uses that idea, one can obtain, instead of the surface of Fig. 4(right), whose top and bottom views are identical, a surface shown in Fig. 6, whose cross-section by a horizontal plane varies with that plane's z-coordinate.

Using the same idea, but different curves, one can generate the Heart symbol in Fig. 2(left) instead of its simplified version of Fig. 3(left).

Lastly, why should the "rotation" be about the z-axis, or, to that end, any straight line? Why not allow the axis of "rotation" swirl (or follow any prescribed 3D curve)? Implementing that idea, one can obtain (among many others) the surface shown in Fig. 7 on the right.

These ideas have been presented to you just FYI. *You do not need to do anything for this section.*

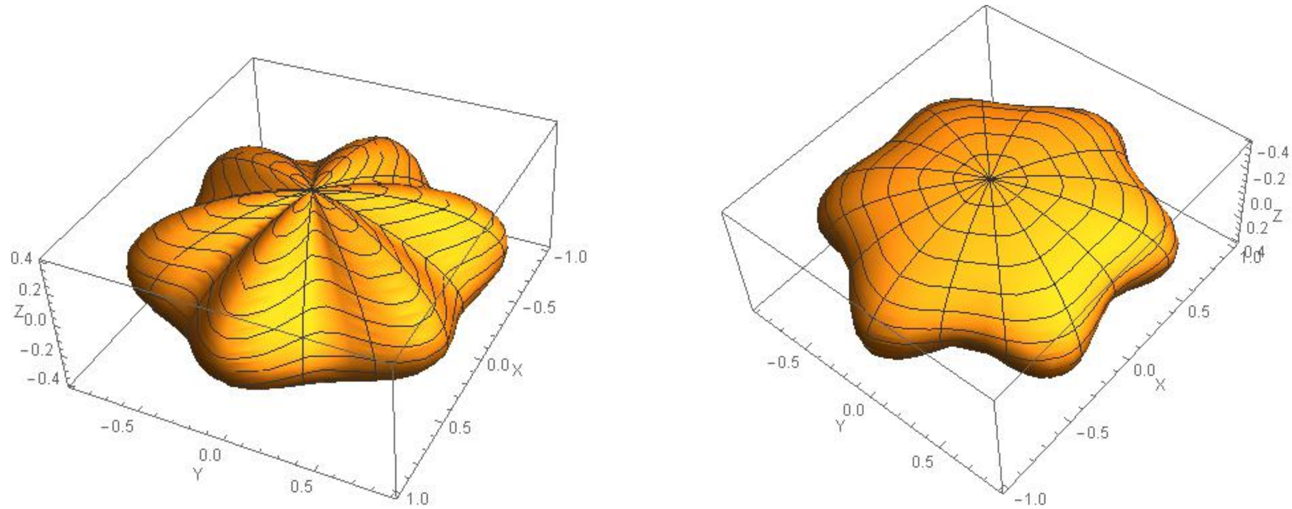


Figure 6: A surface generalizing the one from Fig. 4(right), as explained in the text above. Left: Top view. Right: Bottom view. (Notice that the labels along the z-axis “go” in opposite directions in the left and right panels of this Figure.)

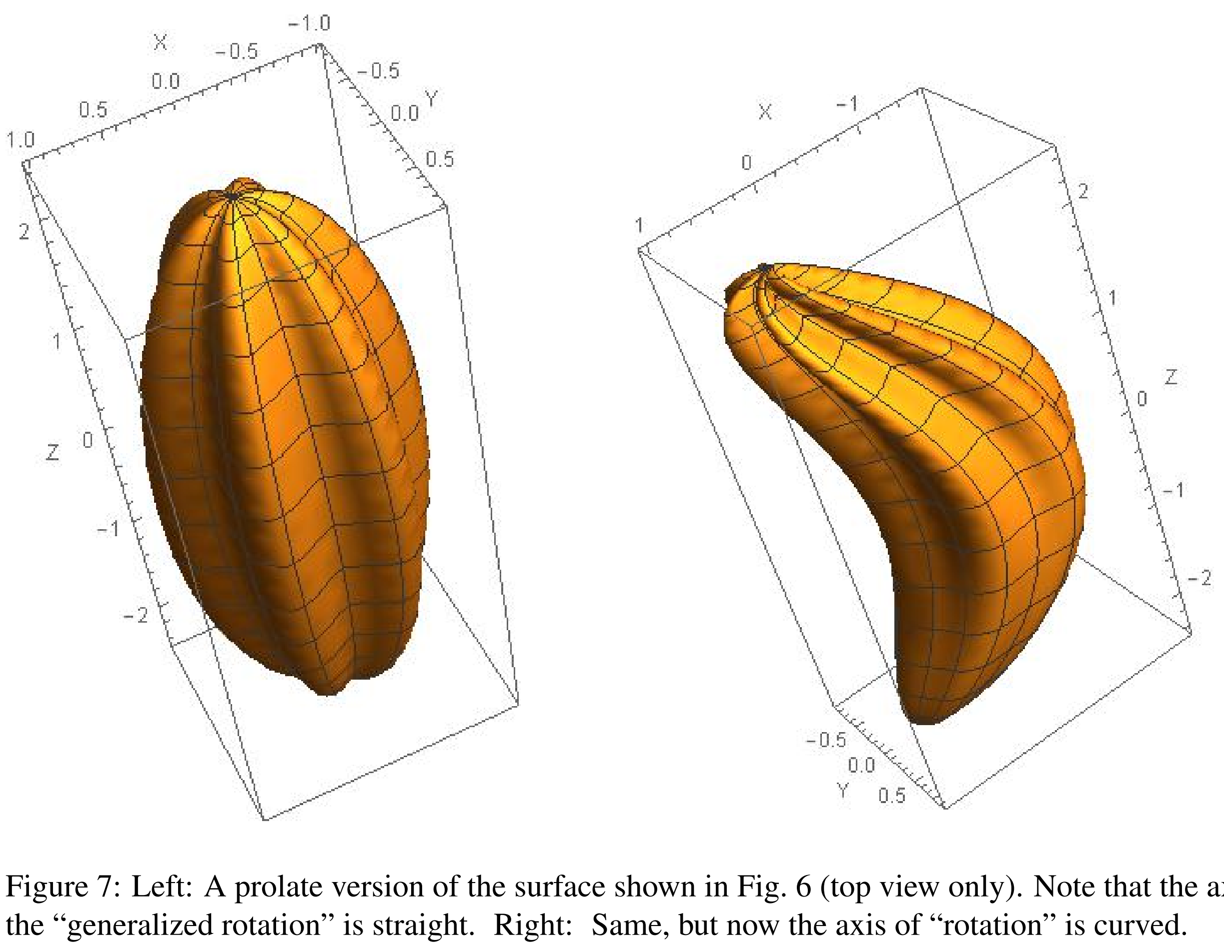


Figure 7: Left: A prolate version of the surface shown in Fig. 6 (top view only). Note that the axis of the “generalized rotation” is straight. Right: Same, but now the axis of “rotation” is curved.